Convergence to Equilibrium and Linear Systems Duality

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Abstract

For a class of interacting particle systems on a countable set, including the so-called linear systems, self-duality has proved to be a strong tool to show longtime convergence to an invariant state ν_θ from a constant initial state θ. This convergence was extended to a class of translation invariant random initial states through the Liggett-Spitzer coupling.

Here we drop the assumption of translation invariance of the initial state. Instead, we assume only that there exists a global density θ in a certain L^2-sense. We use the duality to carry out a comparison argument and show convergence to the same ν_θ as above. We treat some examples in more detail: the parabolic Anderson model, the mutually catalytic branching model, and the smoothing and potlatch processes.

1 Introduction

Our purpose here is to present some new weak convergence results for interacting particle systems that enjoy the "linear systems" type duality as described in Chapter IX of Liggett (1985). This particular form of duality was first exploited by Spitzer

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(1981) in his study of the smoothing, potlatch, and coupled random walk processes. For this class of processes, duality and a simple martingale argument prove weak convergence of the process from constant initial states. To obtain convergence from nonconstant initial states, a coupling method introduced in Liggett and Spitzer (1981) can be used. The coupling method is not duality based.

The Liggett–Spitzer coupling method generally requires that the law of the initial state be translation invariant. We show here, for interacting particle systems with the linear systems type duality, that duality itself can be used to prove weak convergence of the process from some nonconstant initial states. Translation invariance of the initial state is not needed, nor is it necessary to assume translation invariance of the basic mechanism defining the process. We will apply our method to three examples: the parabolic Anderson (or linear random potential) model considered in Shiga (1992), the mutually catalytic branching process introduced in Dawson and Perkins (1998), and the smoothing and potlatch processes.

This work originated in an attempt to extend Theorem 1.4 in Dawson and Perkins (1998) to nonconstant initial states. Since the Liggett–Spitzer coupling does not seem to apply to the mutually catalytic branching process, another method was needed. We found that the model's self–duality property could be used. This duality is similar to the linear systems duality, and we found that our method could also be applied in the linear systems setting.

The heart of our proofs is a convergence in probability statement for the dual process in a fairly general setting (Theorem 3.1). Since this proposition looks rather abstract without some motivation, we start in Section 2 by giving a direct proof of convergence for the parabolic Anderson model, which is the simplest of the models we treat. We then present the general convergence result in Section 3, and use it to obtain convergence results for the mutually catalytic branching process in Section 4 and the smoothing and potlatch processes in Section 5. We note that the method actually gives more than convergence to an equilibrium state from a broad class of initial distributions—it shows that if we pick the intial state from this initial distribution, the resulting random probability measure on the state space converges in probability to the equilibrium measure as time tends to infinity. This extension is used in Cox and Klenke (1998) to show that for a class of processes (including the mutually catalytic branching model in the recurrent setting) one obtains a.s. accumulation at all points in the support of the equilibrium measure.

As will become clear when we give the details of our first example, our method does not provide any new information on the specific nature of a given weak limit. Depending on the parameters involved, such a limit may or may not be "degenerate" (e.g., concentrated on the zero configuration). It is a fundamental problem to determine which is the case. What we show, roughly speaking, is that if two initial states have the same spatial density, measured in an appropriate way, and the process starting in one of these states has a weak limit, then the process starting in the other state has the same weak limit.

We introduce now some notation that will be common to our examples. Let S be a countable set, and let $p(i,j)$, $i,j \in S$ be a (discrete time) irreducible Markov chain transition matrix. We assume p is doubly stochastic so that $\tilde{p}(i,j) = p(j,i)$ is also a transition matrix. Define the continuous time kernel p_t, $t \geq 0$ by

$$p_t(i,j) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!}\, p^{(n)}(i,j), \tag{1.1}$$

where $p^{(n)}$ is the nth iterate of p. For $\phi\colon S \to [0,\infty)$, define $P_t\phi$ and ϕP_t by

$P_t\phi(i) = \sum_{j\in S} p_t(i,j)\phi(j)$ and $\phi P_t(j) = \sum_{j\in S}\phi(i)p_t(i,j)$. For $\phi,\psi\colon S\to[0,\infty)$, let $\langle\phi,\psi\rangle = \sum_{i\in S}\phi(i)\psi(i)$. For $\theta\in[0,\infty)$ let $\boldsymbol{\theta}\in[0,\infty)^S$ be defined by $\boldsymbol{\theta}(j)\equiv\theta$. Let X_F be the set of $x\in[0,\infty)^S$ such that $x(j)=0$ for all but finitely many $j\in S$, and let X_f denote the set of those $x\in[0,\infty)^S$ such that $\langle x,\mathbf{1}\rangle<\infty$. Unless otherwise noted, all sums will be taken over S. Finally, I denotes the identity matrix, $\mathcal{L}$ denotes law, $\overset{\mathrm{d}}{=}$ denotes equality in distribution, and $\Rightarrow$ denotes weak convergence of probability measures.

2 Parabolic Anderson Model

We consider an interacting diffusion x_t, taking values in $[0,\infty)^S$, called the parabolic Anderson model in Shiga (1992). The evolution of x_t is determined by the equation

$$dx_t(i) = (p-I)x_t(i)\,dt + cx_t(i)\,dB_t(i), \qquad i\in S. \tag{2.1}$$

Here c is a fixed positive constant and $\{B_t(i), i\in S\}$ is a collection of independent one–dimensional Brownian motions. To define the state space X, let $\gamma\in[0,\infty)^S$ be a strictly positive, summable reference measure satisfying, for some finite constant Γ,

$$\gamma p \le \Gamma\gamma. \tag{2.2}$$

There is always such a reference measure (see Liggett and Spitzer (1981)). Let $X = X_\gamma = \{x\in[0,\infty)^S : \langle x,\gamma\rangle<\infty\}$. We endow X with the topology generated by componentwise convergence. By results of Shiga and Shimizu (1980), for each starting point, $x_0\in X$, there is a unique strong solution x_t of (2.1) taking values in X. We let $\mathbf{P}^{x_0}$ denote its law on $C([0,\infty),X)$. Note that (x_t) is linear in the following sense: If (x_t^1) and (x_t^2) are solutions of (2.1) with the same (B_t), and a and b are constants, then

$$x_t^3 = x_t^1 + x_t^2 \tag{2.3}$$

is also a solution of (2.1).

As noted in (2.5) of Cox, Greven and Shiga (1995), for $x_0\in X_F$, $E^{x_0}[x_t(i)] = P_tx_0(i)$, and

$$\begin{aligned}&\mathbf{E}^{x_0}[x_t(i)x_t(j)]\\ &\qquad = P_tx_0(i)P_tx_0(j) + c^2\int_0^t \sum_k p_{t-r}(i,k)p_{t-r}(j,k)\mathbf{E}^{x_0}[x_r^2(k)]\,dr.\end{aligned} \tag{2.4}$$

It follows that

$$\mathbf{E}^{x_0}[\langle x_t,\mathbf{1}\rangle^2] = \langle x_0,\mathbf{1}\rangle^2 + c^2\int_0^t \mathbf{E}^{x_0}[\langle x_r^2,\mathbf{1}\rangle]\,dr. \tag{2.5}$$

By standard arguments and Gronwall's inequality, for all $x_0\in X_f$ and $t\ge 0$,

$$\mathbf{E}^{x_0}[\langle x_t,\mathbf{1}\rangle^2]<\infty. \tag{2.6}$$

It is easy to see from (2.1) and (2.6) that for $x_0\in X_f$, $\langle x_t,\mathbf{1}\rangle$ is a continuous, square-integrable martingale.

Let $\tilde{x}_t$ denote the process determined by (2.1), but with $\tilde{p}$ instead of p. Then x_t and $\tilde{x}_t$ are dual in the following sense. Given initial states $x_0 \in X$ and $\tilde{x}_0 \in X_F$,

$$\langle x_t, \tilde{x}_0 \rangle \stackrel{\mathrm{d}}{=} \langle x_0, \tilde{x}_t \rangle. \tag{2.7}$$

A proof of this fact follows exactly as in Theorem IX.1.25 in Liggett (1985).

Suppose now that $\theta > 0$ is fixed, and x_t^θ is the parabolic Anderson model with initial state $x_0^\theta = \boldsymbol{\theta}$. Then, for any $\phi \in X_F$, (2.7) implies that

$$\langle x_t^{\theta}, \phi \rangle \stackrel{\mathrm{d}}{=} \theta \langle \mathbf{1}, \tilde{x}_t \rangle, \qquad \tilde{x}_0 = \phi. \tag{2.8}$$

The right side of (2.8) is a nonnegative martingale, and hence converges a.s. as $t \to \infty$. Therefore, the left side of (2.8) must converge as $t \to \infty$. On account of this, there is a probability measure ν_θ on X such that

$$\mathcal{L}[x_t^{\boldsymbol{\theta}}] \Rightarrow \nu_\theta \qquad \text{as } t \to \infty. \tag{2.9}$$

In particular,

$$\begin{aligned} \mathbf{E}^\phi[e^{-\theta\langle \mathbf{1}, \tilde{x}_t\rangle}] &= \mathbf{E}[e^{-\langle x_t^\theta, \phi\rangle}] \\ &\to \int e^{-\langle x, \phi\rangle}\, \nu_\theta(dx) \qquad \text{as } t \to \infty. \end{aligned} \tag{2.10}$$

Note that by the linearity property (2.3), $\nu_\theta(A) = \nu_1(\theta^{-1}A)$.

In some cases, the limit ν_θ may be concentrated on the configuration which is identically 0. In Remark 2.2 below we recall conditions which determine, in some cases, whether or not this happens. Our goal is to show that convergence to ν_θ holds for a large class of initial states with an appropriately defined spatial density θ.

For $\theta \in [0, \infty)$, define $\mathcal{M}_\theta$ to be the collection of probability measures ν on X such that

$$\sup_k \int x^2(k) d\nu(x) < \infty, \tag{2.11}$$

$$\lim_{t\to\infty} \int (P_t x(k) - \theta)^2 d\nu(x) = 0, \quad k \in S. \tag{2.12}$$

Suppose, for example, P_t is the semigroup of simple symmetric random walk on $\mathbb{Z}^d$. If $\{x(k) : k \in \mathbb{Z}^d\}$ are iid non-negative random variables with mean θ and finite variance, then their law ν is in $\mathcal{M}_\theta$. However, there are a number of non-translation invariant laws in $\mathcal{M}_\theta$. If $x(k) = 1(k_1 < 0)$ clearly $\delta_x \in \mathcal{M}_{1/2}$. More generally, if $x \in [0, \infty)^{\mathbb{Z}^d}$ is bounded and the average value of x in a ball (Euclidean metric) approaches θ as the radius approaches ∞, then it is not hard to show that $\delta_x \in \mathcal{M}_\theta$.

Our main technical result is the following.

Proposition 2.1 *Assume p is doubly stochastic. Let $\theta \in [0, \infty)$ and let $\nu \in \mathcal{M}_\theta$. If x_0 has law ν, and $\tilde{x}_0 \in X_F$, then*

$$\langle x_0 - \boldsymbol{\theta}, \tilde{x}_t \rangle \to 0 \tag{2.13}$$

in $\nu \otimes \mathbf{P}^{\tilde{x}_0}$-probability as $t \to \infty$.

Corollary 2.1 *If $S = \mathbb{Z}^d$, $p(i, j) = p(0, j - i)$, $\nu = \mathcal{L}[x_0]$ is translation invariant, shift ergodic, and satisfies $\int x(0)\, d\nu(x) = \theta$, then (2.13) holds.*

Proof. For $N > 0$ define $x_0^N(i) = x_0(i) \wedge N$, and let $\theta_N = \int [x(0) \wedge N]\, d\nu(x)$. Then

$$\langle x_0 - \boldsymbol{\theta}, \tilde{x}_t \rangle = \langle x_0 - x_0^N, \tilde{x}_t \rangle + \langle x_0^N - \boldsymbol{\theta}_N, \tilde{x}_t \rangle + \langle \boldsymbol{\theta}_N - \boldsymbol{\theta}, \tilde{x}_t \rangle.$$

It is easy to see (Theorem 5.6 in Liggett 1973) that $\mathcal{L}[x_0^N] \in \mathcal{M}_{\theta_N}$, so that for each N, $\langle x_0^N - \boldsymbol{\theta}_N, \tilde{x}_t \rangle \to 0$ in $\nu \otimes \mathbf{P}^{\tilde{x}_0}$-probability as $t \to \infty$. On the other hand, letting $\mathbf{E}$ denote expectation with respect to $\nu \otimes \mathbf{P}^{\tilde{x}_0}$,

$$\mathbf{E}[\langle x_0 - x_0^N, \tilde{x}_t \rangle] \leq \int x(0) 1_{\{x(0)>N\}}\, d\nu(x)\ \langle \mathbf{1}, \tilde{x}_0 \rangle \to 0$$

and

$$\mathbf{E}[\langle \boldsymbol{\theta} - \boldsymbol{\theta}_N, \tilde{x}_t \rangle] = (\theta - \theta_N)\ \langle \mathbf{1}, \tilde{x}_0 \rangle \to 0$$

as $N \to \infty$. $\square$

Let us see now what duality and (2.13) imply. Let $\tilde{x}_0 \in X_F$, and let x_0 have law ν which satisfies (2.13). By (2.7),

$$\begin{aligned} \mathbf{E}[e^{-\langle x_t, \tilde{x}_0 \rangle}] &= \mathbf{E}^\nu \otimes \mathbf{E}^{\tilde{x}_0}[e^{-\langle x_0, \tilde{x}_t \rangle}] \\ &= \mathbf{E}^\nu \otimes \mathbf{E}^{\tilde{x}_0}[e^{-\langle x_0 - \boldsymbol{\theta}, \tilde{x}_t \rangle}\ e^{-\theta \langle \mathbf{1}, \tilde{x}_t \rangle}]\,. \end{aligned} \tag{2.14}$$

By (2.13) $e^{-\langle x_0 - \boldsymbol{\theta}, \tilde{x}_t \rangle} \to 1$ in $\mathbf{P}^\nu \otimes \mathbf{P}^{\tilde{x}_0}$-probability as $t \to \infty$. Therefore, in view of (2.10),

$$\lim_{t \to \infty} \mathbf{E}^\nu[e^{-\langle x_t, \phi \rangle}] = \int e^{-\langle x, \phi \rangle}\, d\nu_\theta(x), \tag{2.15}$$

and we have established the following.

Theorem 2.1 *Assume that p is doubly stochastic, and either* (i) $\mathcal{L}[x_0] \in \mathcal{M}_\theta$, *or* (ii) $S = \mathbb{Z}^d$, $p(i,j) = p(0, j-i)$, *and $\mathcal{L}[x_0]$ is translation invariant, shift ergodic, and satisfies $\mathbf{E}[x_0(0)] = \theta$. Then $\mathcal{L}[x_t] \Rightarrow \nu_\theta$ as $t \to \infty$.*

Remark 2.1 *It is possible to formulate and prove a stronger type of convergence than given above, which is used in Cox and Klenke (1998). We do this for the mutually catalytic branching process in Theorem 4.1(a) below.*

Remark 2.2 *Suppose that $S = \mathbb{Z}^d$ and $p(i,j) = p(0, j-i)$. If the symmetrization of p is transient, and c is sufficiently small, Theorem 1.1 of Shiga (1992) implies that $\mathcal{L}[x_t] \Rightarrow \nu_\theta$ for any initial law ν which is translation invariant, shift ergodic, and satisfies $\int x(0)\, d\nu(x) = \theta$. If the symmetrization of p is recurrent, Theorem 2 of Cox, Fleischmann and Greven (1996) implies that $\nu_\theta = \delta_{\mathbf{0}}$, and that $\mathcal{L}[x_t] \Rightarrow \delta_{\mathbf{0}}$ for any initial law ν which is translation invariant and satisfies $\int x(0)\, d\nu(x) < \infty$. For information concerning the case when the symmetrization of p is transient and c is large, see Theorem 1.2 in Shiga (1992).*

Proof of Proposition 2.1. Recall that the total mass process $\langle \tilde{x}_t, \mathbf{1} \rangle$ is a continuous square-integrable martingale. By (2.1) and (2.6), $\langle \tilde{x}_t, \mathbf{1} \rangle = \langle \tilde{x}_0, \mathbf{1} \rangle + \sum_i M_t(i)$, where

$$M_t(i) = c \int_0^t \tilde{x}_s(i) dB_s(i)$$

and the sum converges in L^2. The $M_t(i)$ are continuous, square-integrable orthogonal martingales, with

$$\langle M(i)\rangle_t = c^2\int_0^t \tilde{x}_s^2(i)\,ds.$$

The continuous martingale $\langle \tilde{x}_t, \mathbf{1}\rangle$ has integrable square function $A_t = \sum_i \langle M(i)\rangle_t$. Since $\langle \tilde{x}_t, \mathbf{1}\rangle$ converges a.s. $\mathbf{P}^{\tilde{x}_0}$ as $t\to\infty$,

$$\begin{aligned} A_\infty &= \sum_i \langle M(i)\rangle_\infty \\ &= c^2\int_0^\infty \sum_i \tilde{x}_s(i)^2\,ds < \infty \qquad \text{a.s. } \mathbf{P}^{\tilde{x}_0}. \end{aligned} \tag{2.16}$$

It is straightforward to check, by applying Itô's formula to $\tilde{P}_{t-s}\tilde{x}_s(i)$, that we have the representation

$$\tilde{x}_t(i) = \tilde{P}_t\tilde{x}_0(i) + \int_0^t \sum_j \tilde{p}_{t-s}(i,j)\,dM_s(j). \tag{2.17}$$

Let x_0 have law ν, let $\Delta = x_0 - \boldsymbol{\theta}$, and set $\mathbf{P} = \nu\otimes\mathbf{P}^{\tilde{x}_0}$. Since $\nu\in\mathcal{M}_\theta$, $C = \sup_i \int[\Delta(i)^2]d\nu$ is finite. It follows easily that

$$\sup_{i,t}\int \tilde{P}_t\Delta(i)^2 d\nu \le C, \tag{2.18}$$

and also that ($\mathbf{E}$ denotes expectation with respect to $\mathbf{P}$)

$$\mathbf{E}\Big[\sum_i \int_0^t \tilde{P}_{t-s}\Delta(i)^2\tilde{x}_s(i)^2\,ds\Big] \le C\int_0^t \mathbf{E}^{\tilde{x}_0}[\langle \tilde{x}_s, \mathbf{1}\rangle^2]\,ds < \infty. \tag{2.19}$$

Consequently, letting

$$N_s^t = \sum_i \int_0^s \Delta\tilde{P}_{t-r}(i)\,dM_r(i), \qquad s\le t, \tag{2.20}$$

we have

$$\langle \Delta, \tilde{x}_t\rangle = \langle \Delta, \tilde{P}_t\tilde{x}_0\rangle + N_t^t. \tag{2.21}$$

Here, $\{N_s^t, s\le t\}$ is a continuous, square-integrable martingale under $\mathbf{P}$, with square variation function

$$\langle N^t\rangle_s = \int_0^s \sum_i \Delta\tilde{P}_{t-r}(i)^2\,d\langle M(i)\rangle_r < \infty \quad \text{a.s. } \mathbf{P}^{\tilde{x}_0}.$$

Here we have added x_0 to the underlying filtration at time 0.

We are now ready to prove (2.13). It is straightforward to check that

$$\begin{aligned} \int \langle \Delta, \tilde{P}_t\tilde{x}_0\rangle^2 d\nu &= \int \langle P_t\Delta, \tilde{x}_0\rangle^2 d\nu \\ &= \sum_{j,k} \tilde{x}_0(j)\tilde{x}_0(k)\int P_t\Delta(j)P_t\Delta(k)\,d\nu \\ &\le \Big[\sum_j \tilde{x}_0(j)\Big(\int P_t\Delta(j)^2\,d\nu\Big)^{1/2}\Big]^2. \end{aligned}$$

Using (2.12), (2.18), and the fact that $\tilde{x}_0 \in X_f$, it follows that

$$\int \langle \Delta, \tilde{P}_t \tilde{x}_0 \rangle^2 d\nu \to 0 \quad \text{as } t \to \infty. \tag{2.22}$$

The next step is to show that for all $\varepsilon' > 0$,

$$\lim_{t\to\infty} \mathbf{P}[\langle N^t \rangle_t > \varepsilon'] = 0. \tag{2.23}$$

To do this, we consider the expectation

$$\begin{aligned}
\int \langle N^t \rangle_t \, d\nu &= \int_0^t \sum_i \int \tilde{P}_{t-r}\Delta(i)^2 \, d\nu \, d\langle M(i) \rangle_r \\
&= \sum_i \int_0^\infty \mathbf{1}_{\{r<t\}} \int \tilde{P}_{t-r}\Delta(i)^2 \, d\nu \, d\langle M(i) \rangle_r .
\end{aligned}$$

We note that $\mathbf{1}_{\{r<t\}} \int \tilde{P}_{t-r}\Delta(i)^2 \, d\nu \to 0$ as $t \to \infty$, and is bounded by C (by (2.18)). Hence, on account of (2.16) and the bounded convergence theorem,

$$\mathbf{P}^{\tilde{x}_0}\left[\lim_{t\to\infty} \int \langle N^t \rangle_t \, d\nu = 0\right] = 1. \tag{2.24}$$

Therefore,

$$\mathbf{P}[\langle N^t \rangle_t > \varepsilon'] = \mathbf{E}^{\tilde{x}_0}\left[\nu[\langle N^t \rangle_t > \varepsilon']\right] \to 0 \tag{2.25}$$

as $t \to \infty$, which proves (2.23) .

By a standard stopping time argument for martingales, for any $\varepsilon > 0$,

$$\mathbf{P}[|N_t^t| > \varepsilon, \langle N^t \rangle_t \le \varepsilon^3] \le \varepsilon. \tag{2.26}$$

The claim (2.13) now follows from (2.21)– (2.23) and (2.26) (set $\varepsilon' = \varepsilon^3$). $\square$

3 A Convergence Result for the Dual Process

The key step in the proof of the convergence to equilibrium for the parabolic Anderson model was the convergence in probability statement for the dual process given in Proposition 2.1. As the argument leading to that conclusion applies in a number of different settings exhibiting a similar duality, we now prove a general result. Recall that P_t is the continuous time transition matrix associated with a doubly stochastic matrix $p(i,j)$ on a countable set S. Let $\{S_n\}$ be a sequence of finite sets which increase to S as $n \to \infty$. We endow the set $[0,\infty)^S$ with the topology of pointwise convergence.

Let $\{x_t(i) : t \ge 0, i \in S\}$ be a right-continuous with left limits (RCLL) stochastic process taking values in $[0,\infty)^S$, defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P}^x)$. Assume

$$X_t \equiv \langle x_t, \mathbf{1} \rangle \text{ is a square-integrable RCLL martingale,} \tag{3.1}$$

and there is a family of RCLL L^2-martingales $\{M_t(i), t \ge 0, i \in S\}$ such that, a.s. for all $i \in S$ and $t \ge 0$,

$$x_t(i) = P_t x_0(i) + \sum_j \int_0^t P_{t-s}(i,j) \, dM_s(j) = P_t x_0(i) + \left[\int_0^t P_{t-s} \, dM_s\right](i). \tag{3.2}$$

Here it is understood that the series in (3.2) converges in L^2. For $\phi : S \to \mathbb{R}$ let

$$Q_t(\phi) = \sum_{i,j} \phi(i)\phi(j)\langle M(i), M(j)\rangle_t, \tag{3.3}$$

$$|Q|_t(\phi) = \sum_{i,j} \phi(i)\phi(j)|\langle M(i), M(j)\rangle|_t, \tag{3.4}$$

where $|\langle M(i), M(j)\rangle|_t$ is the total variation of $\langle M(i), M(j)\rangle$ up to time t. We will also use the notation

$$\int_0^t \phi\, dQ_t = \int_0^t \sum_{i,j} \phi(i)\phi(j)\, d\langle M(i), M(j)\rangle_t,$$

$$\int_0^t \phi\, d|Q|_t = \int_0^t \sum_{i,j} \phi(i)\phi(j)\, d|\langle M(i), M(j)\rangle|_t.$$

To ensure these expressions make sense, at least for bounded ϕ, we assume

$$\mathbf{E}[|Q|_t(\mathbf{1})] < \infty \qquad \text{for all } t \geq 0. \tag{3.5}$$

Here is our main convergence result.

Theorem 3.1 *In addition to* (3.1)–(3.5)*, assume that*

$$|Q|_\infty(\mathbf{1}) < \infty \quad a.s. \tag{3.6}$$

Let ν be a probability measure on $\mathbb{R}^S$ such that

$$\begin{aligned} C(\nu) &\equiv \sup_{k\in S} \int \phi(k)^2\, d\nu(\phi) < \infty, \\ \lim_{t\to\infty} &\int (\phi P_t(k))^2\, d\nu(\phi) = 0 \qquad \textit{for all } k \in S. \end{aligned} \tag{3.7}$$

Then $\langle \phi, x_t\rangle \to 0$ in $\nu \otimes \mathbf{P}^x$-probability as $t \to \infty$.

Proof. Let $\mathbf{P} = \nu \otimes \mathbf{P}^x$, and let (ϕ, ω) denote a generic point in $\mathbb{R}^S \times \Omega$. Let $N_s^t(i) = \left[\int_0^s P_{t-r} dM_r\right](i)$, $s \leq t$, so that we may write $x_t = P_t x_0 + N_t^t$. By (3.2),

$$\langle \phi \mathbf{1}_{S_n}, x_t\rangle = \langle \phi \mathbf{1}_{S_n}, P_t x_0\rangle + \langle \phi \mathbf{1}_{S_n}, N_t^t\rangle.$$

As $n \to \infty$, the left side and first term on the right side above converge in $L^2(\mathbf{P})$ to $\langle \phi, x_t\rangle$ and $\langle \phi, P_t x_0\rangle$, respectively. This is because, for $m < n$,

$$\begin{aligned} \mathbf{E}\Big[\langle \phi(\mathbf{1}_{S_n} - \mathbf{1}_{S_m}), x_t\rangle^2 &+ \langle \phi(\mathbf{1}_{S_n} - \mathbf{1}_{S_m}), P_t x_0\rangle^2\Big] \\ &\leq C(\nu)\mathbf{E}^x\Big[\langle \mathbf{1}_{S_n} - \mathbf{1}_{S_m}, x_t\rangle^2 + \langle \mathbf{1}_{S_n} - \mathbf{1}_{S_m}, P_t x_0\rangle^2\Big], \end{aligned}$$

and by (3.1), the right side above tends to 0 as $m, n \to \infty$. Therefore,

$$\langle \phi, N_t^t\rangle = \lim_{n\to\infty} \langle \phi \mathbf{1}_{S_n}, N_t^t\rangle \qquad \text{in } L^2$$

exists, and by Doob's strong L^2 inequality, $\mathbf{E}[\sup_{s\leq t} |\langle \phi 1_{S_n}, N_s^t\rangle - \langle \phi, N_s^t\rangle|^2] \to 0$ as $n \to \infty$. Consequently, $\langle \phi, N_s^t\rangle$ is a RCLL L^2-martingale, and its predictable square function is the L^2-limit of $\sum_{j,j'} \int_0^s (\phi \mathbf{1}_{S_n} P_{t-r})(j)(\phi \mathbf{1}_{S_n} P_{t-r})(j') d\langle M(j), M(j')\rangle_r$, given by

$$\sum_{j,j'} \int_0^s \phi P_{t-r}(j) \phi P_{t-r}(j') d\langle M(j), M(j')\rangle_r = \int_0^s (\phi P_{t-r})\, dQ_r. \tag{3.8}$$

(The L^2 limit follows easily from (3.5), (3.7) and the dominated convergence theorem.) To sum up, we have established that

$$\langle \phi, x_t\rangle = \langle \phi, P_t x_0\rangle + \langle \phi, N_t^t\rangle, \tag{3.9}$$

where $\{\langle \phi, N_s^t\rangle, s \leq t\}$ is a RCCL L^2-martingale, with predictable square function $\langle\langle \phi, N^t\rangle\rangle_s = \int_0^s (\phi P_{t-r})\, dQ_r$.

We must show both terms on the right side of (3.9) tend to 0 in $\mathbf{P}$-probability as $t \to \infty$. The first term is easy. By (3.7),

$$\sup_{k \in S} \int \phi P_t(k)^2 \, d\nu(\phi) \leq C(\nu)\,. \tag{3.10}$$

Therefore, we have

$$\begin{aligned}
\int \langle \phi, P_t x_0\rangle^2 \, d\nu(\phi) &= \int \langle \phi P_t, x_0\rangle^2 \, d\nu(\phi) \\
&= \sum_{j,k} x_0(j) x_0(k) \int (\phi P_t)(j)(\phi P_t)(k) \, d\nu(\phi) \\
&\leq \Big[\sum_j x_0(j) \Big(\int (\phi P_t(j))^2 \, d\nu(\phi)\Big)^{1/2}\Big]^2 \\
&\to 0
\end{aligned} \tag{3.11}$$

as $t \to \infty$ by (3.1), (3.7) and dominated convergence.

To handle the martingale term in (3.9), we have

$$\begin{aligned}
\int \langle\langle \phi, N^t\rangle\rangle_t \, d\nu(\phi) &= \int \int_0^t (\phi P_{t-r}) dQ_r \, d\nu(\phi) \\
&\leq \int_0^\infty \mathbf{1}_{\{r<t\}} \Big(\int (\phi P_{t-r})^2 \, d\nu(\phi)\Big)^{1/2} \, d|Q|_r.
\end{aligned}$$

Note that $\mathbf{1}_{\{r<t\}}[\int (\phi P_{t-r}(k))^2 \, d\nu(\phi)]^{1/2} \to 0$ pointwise as $t \to \infty$ by (3.7), and by (3.10) is bounded by $C(\nu)^{1/2}$. Therefore, by (3.6) and dominated convergence,

$$\lim_{t \to \infty} \int \langle\langle \phi, N^t\rangle\rangle_t \, d\nu(\phi) = 0 \qquad \mathbf{P}^x\text{-a.s.} \tag{3.12}$$

Using this fact and a standard stopping time argument for martingales, we have

$$\begin{aligned}
\mathbf{P}[|\langle \phi, N_t^t\rangle| > \varepsilon] &\leq \mathbf{P}[\langle\langle \phi, N^t\rangle\rangle_t > \varepsilon^3] + \mathbf{P}[|\langle \phi, N_t^t\rangle| > \varepsilon, \langle\langle \phi, N^t\rangle\rangle_t \leq \varepsilon^3] \\
&\leq \mathbf{E}^x \left[\nu(\langle\langle \phi, N^t\rangle\rangle_t > \varepsilon^3)\right] + \varepsilon \\
&< 2\varepsilon
\end{aligned}$$

for t large. This completes our proof. $\square$

For the applications we have in mind, (3.1)–(3.5) will be easy to verify and (3.7) will be our working hypothesis on ν. We now assume (3.1)–(3.5) and provide some alternative conditions which imply (3.6).

If we repeat the first part of the previous proof with $\phi = \mathbf{1}$, we see that

$$\langle x_t, \mathbf{1}\rangle = \langle x_0, \mathbf{1}\rangle + \sum_i \Big[\int_0^t P_{t-s}\, dM_s\Big](i),$$

where the sum converges a.s. and in L^2. Let $P_r(S_n, j) = \sum_{i\in S_n} p_r(i,j)$. Then

$$\begin{aligned}\sum_{i\in S_n}\Big[\int_0^t P_{t-s} dM_s\Big](i) &= \sum_{j\in S_m}\int_0^t P_{t-s}(S_n, j)\, dM_s(j) + \sum_{j\in S_m^c}\int_0^t P_{t-s}(S_n,j)\, dM_s(j)\\ &\equiv \Sigma^1_{n,m} + \Sigma^2_{n,m}.\end{aligned}$$

Dominated convergence and (3.5) imply that

$$\lim_{n\to\infty} \mathbf{E}^{x_0}\Big[|\Sigma^1_{n,m} - \sum_{j\in S_m} M_t(j)|^2\Big] = 0, \qquad \lim_{m\to\infty}\sup_n \mathbf{E}^{x_0}\left[|\Sigma^2_{m,n}|^2\right] = 0.$$

This easily shows that $\sum_j M_t(j)$ converges in L^2 to $\sum_i [\int_0^t P_{t-s} dM_s(i)]$, and so (recall (3.1))

$$X_t = X_0 + \sum_j M_t(j). \tag{3.13}$$

On account of this,

$$\langle X\rangle_t = Q_t(\mathbf{1}) \qquad \text{for all } t \geq 0 \ a.s. \tag{3.14}$$

Now, by the martingale convergence theorem, $X_\infty = \lim_{t\to\infty} X_t$ exists and is finite a.s., and we would like to infer that $Q_\infty(\mathbf{1}) = \lim_{t\to\infty} Q_t(\mathbf{1})$ is finite a.s. too, but this requires an additional hypothesis if X is not continuous.

Lemma 3.1 *Assume $(N_t, t \geq 0)$ is a nonnegative, square-integrable, RCLL martingale. If $T_n = \inf\{t : N_t \geq n\}$ (with $\inf\emptyset = \infty$), and for each positive integer n, $\mathbf{E}[(N_{T_n} - N_{T_n-})^2 1_{\{T_n<\infty\}})] < \infty$, then $\lim_{t\to\infty}\langle N\rangle_t < \infty$ a.s.*

Proof. The assumption on the jumps implies that, for each n, the martingale $M_t^{(n)} = N^2_{t\wedge T_n} - \langle N\rangle_{t\wedge T_n}$ is L^1-bounded. Hence, as $t \to \infty$, $M_t^{(n)}$ converges a.s. to a finite limit $M_\infty^{(n)}$. Since N_t is a nonnegative martingale, N_t converges a.s. to a finite limit N_∞. Therefore, on $\{T_n = \infty\}$, $\langle N\rangle_t \to \langle N\rangle_\infty = N_\infty^2 - M_\infty^{(n)}$ as $t \to \infty$. The fact that N_∞ is finite a.s. implies that $\mathbf{P}[\cup_n\{T_n = \infty\}] = 1$, and hence $\mathbf{P}[\langle N\rangle_\infty < \infty] = 1$. $\square$

Here then are two conditions which together will imply (3.6). The first one is

$$d\langle M(i), M(j)\rangle_t \geq 0 \qquad \text{for all } i,j\in S, t\geq 0 \ a.s. \tag{3.15}$$

The second, with $T_n = \inf\{t : \langle x_t, \mathbf{1}\rangle \geq n\}$, is

$$\mathbf{E}[(\langle x_{T_n}, \mathbf{1}\rangle - \langle x_{T_n-}, \mathbf{1}\rangle)^2 1_{\{T_n<\infty\}}] < \infty \quad \text{for all } n. \tag{3.16}$$

Theorem 3.2 *Assume* (3.1)–(3.5), (3.15) *and* (3.16). *Then* (3.6) *holds, and so if ν is a probability measure on $\mathbb{R}^S$ satisfying* (3.7), *then $\langle\phi, x_t\rangle \to 0$ in $\nu\otimes\mathbf{P}^{x_0}$-probability as $t\to\infty$.*

Proof. Lemma 3.1 implies $Q_\infty(\mathbf{1}) < \infty$ a.s., and (3.15) implies $|Q|_\infty(\mathbf{1}) = Q_\infty(\mathbf{1})$ a.s. Therefore, (3.6) is true, and Theorem 3.1 completes the proof. $\square$

4 Mutually Catalytic Branching

As in Dawson and Perkins (1998), let (u_t, v_t) denote the mutually catalytic branching model defined by

$$\begin{aligned} du_t &= (p-I)u_t\,dt + (cu_tv_t)^{1/2}dB_t, \\ dv_t &= (p-I)v_t\,dt + (cu_tv_t)^{1/2}dW_t. \end{aligned} \tag{4.1}$$

Here, $p(i,j)$ is an irreducible Markov chain transition matrix, c is a fixed positive constant, and the $\{B_t(i)\}$ and $\{W_t(i)\}$ are independent families of independent one–dimensional standard Brownian motions. As in Dawson and Perkins (1998), we assume $S = \mathbb{Z}^d$, $p(i,j)$ is symmetric, and the exponential growth condition (H2) of that paper holds. This growth condition is satisfied, for example, when $p(i,j)$ is the transition function of a symmetric random walk such that $\sum_k \phi_{-\lambda}(k)p(0,k) < \infty$ for all $\lambda > 0$, where $\phi_\lambda(k) = e^{\lambda|k|}$, $|k| = \sum_{i=1}^d |k_i|$. To define an appropriate state space, we introduce

$$M_{\text{tem}} = \big\{u\colon \mathbb{Z}^d \to \mathbb{R}_+ \text{ such that } \langle u, \phi_\lambda\rangle < \infty \quad \forall \lambda < 0\big\}.$$

Let $|u|_\lambda = \sup\{|u(k)|\phi_\lambda(k) : k \in \mathbb{Z}^d\}$, and topologize M_{tem} so that $u_n \to u$ in M_{tem} iff $\lim_{n\to\infty} |u_n - u|_\lambda = 0$ for all $\lambda < 0$. For each pair of initial conditions $(u_0, v_0) \in M_{\text{tem}}^2$ there is a well defined Markov process (u_t, v_t) determined by (4.1) taking values in M_{tem}^2 (see Theorems 1.1 and 2.4 in Dawson and Perkins (1998) for precise details), and its law $\mathbf{P}^{(u_0,v_0)}$ on $C([0,\infty), M_{\text{tem}}^2)$ is unique.

The mutually catalytic branching process (u_t, v_t) is self–dual. Let (u_t, v_t) be the mutually catalytic branching process with initial state (u_0, v_0), and let $(\tilde u_t, \tilde v_t)$ be the mutually catalytic branching process with initial state $(\tilde u_0, \tilde v_0)$. Mytnik (1998) (see also Theorem 2.4 in Dawson and Perkins (1998)) showed that for $u_0, v_0 \in M_{\text{tem}}$ and $\tilde u_0, \tilde v_0 \in X_F$,

$$\begin{aligned} &\mathbf{E}^{(u_0,v_0)}\big[e^{-\langle u_t+v_t, \tilde u_0+\tilde v_0\rangle + i\langle u_t - v_t, \tilde u_0 - \tilde v_0\rangle}\big] \\ &\qquad = \mathbf{E}^{(\tilde u_0,\tilde v_0)}\big[e^{-\langle u_0+v_0, \tilde u_t+\tilde v_t\rangle + i\langle u_0 - v_0, \tilde u_t - \tilde v_t\rangle}\big]. \end{aligned} \tag{4.2}$$

Suppose now that $a, b \geq 0$ are fixed. Let (u_t^a, v_t^b) denote the mutually catalytic branching model with initial state $(u_0^a, v_0^b) = (\mathbf{a}, \mathbf{b})$. Then (4.2) implies that for $\tilde u_0$, $\tilde v_0$ in X_F,

$$\begin{aligned} &\mathbf{E}\big[e^{-\langle u_t^a+v_t^b, \tilde u_0+\tilde v_0\rangle + i\langle u_t^a - v_t^b, \tilde u_0 - \tilde v_0\rangle}\big] \\ &\qquad = \mathbf{E}^{(\tilde u_0,\tilde v_0)}\big[e^{-(a+b)\langle \mathbf{1}, \tilde u_t+\tilde v_t\rangle + i(a-b)\langle \mathbf{1}, \tilde u_t - \tilde v_t\rangle}\big]. \end{aligned} \tag{4.3}$$

By Theorem 2.2 in Dawson and Perkins (1998), $\langle \mathbf{1}, \tilde u_t\rangle$ and $\langle \mathbf{1}, \tilde v_t\rangle$ are nonnegative martingales, and hence converge a.s. $\mathbf{P}^{(\tilde u_0,\tilde v_0)}$ as $t \to \infty$. Therefore, the left side of (4.3) must converge in distribution as $t \to \infty$. It is not hard to see that this implies there is a stationary probability measure $\nu_{(a,b)}$ on M_{tem}^2 such that

$$\mathcal{L}[(u_t^a, v_t^b)] \Rightarrow \nu_{(a,b)} \qquad \text{as } t \to \infty$$

in the sense of weak convergence of probabilities on M_{tem}^2 (see Theorem 1.4 of Dawson and Perkins (1998)). In particular,

$$\begin{aligned} \mathbf{E}^{(\phi,\psi)}\big[e^{-(a+b)\langle \mathbf{1}, \tilde u_t+\tilde v_t\rangle + i(a-b)\langle \mathbf{1}, \tilde u_t - \tilde v_t\rangle}\big] &= \mathbf{E}\big[e^{-\langle u_t^a+v_t^b, \phi+\psi\rangle + i\langle u_t^a - v_t^b, \phi-\psi\rangle}\big] \\ &\to \int e^{-\langle u'+v', \phi+\psi\rangle + i\langle u'-v', \phi-\psi\rangle} d\nu_{(a,b)}(u', v') \end{aligned} \tag{4.4}$$

as $t \to \infty$. If T is the first exit time of planar Brownian motion (B_t^1, B_t^2) from the first quadrant starting at (a, b), then under appropriate recurrence hypotheses on P_t (satisfied, for example, by simple symmetric random walk in 1 or 2 dimensions), $\nu_{(a,b)}(\cdot) = \mathbf{P}[(\mathbf{B}_T^1, \mathbf{B}_T^2) \in \cdot]$, where $\mathbf{B}_T^i(k) = B_T^i$ for all k. Hence $u \equiv 0$ or $v \equiv 0$ $\nu_{(a,b)}$-a.s. For transient P_t, $u(k)v(k) > 0$ for all k $\nu_{(a,b)}$-a.s. See Theorems 1.5 and 1.6 of Dawson and Perkins (1998) for the precise results and further information about these limiting laws.

For $a, b \geq 0$, define $\mathcal{M}_{(a,b)}$ to be the collection of probability measures ν on $M_{\text{tem}} \times M_{\text{tem}}$ such that

$$\sup_k \int (u^2(k) + v^2(k))\, d\nu(u, v) < \infty, \tag{4.5}$$

$$\lim_{t\to\infty} \int [(P_t u(k) - a)^2 + (P_t v(k) - b)^2]\, d\nu(u, v) = 0, \quad k \in \mathbb{Z}^d. \tag{4.6}$$

As does $\mathcal{M}_\theta$ in Section 2, this class contains non-translation invariant laws. Assume, for example, that P_t is the semigroup of simple symmetric random walk. If

$$(u_0, v_0) = (1(k_1 < 0), 1(k_1 > 0)), \tag{4.7}$$

then $\delta_{(u_0,v_0)} \in \mathcal{M}_{(1/2,1/2)}$. As for the parabolic Anderson model, if u_0, v_0 are bounded non-negative maps on $\mathbb{Z}^d$ whose averages over Euclidean balls approach a and b, respectively, as the radius of the ball approaches ∞, then $\delta_{(u_0,v_0)} \in \mathcal{M}_{(a,b)}$.

Our main technical result for the mutually catalytic branching model is:

Proposition 4.1 *Let $a, b \geq 0$, and $\nu \in \mathcal{M}_{(a,b)}$. If (u_0, v_0) has law ν, and $(\tilde{u}_0, \tilde{v}_0) \in X_F \times X_F$, then*

$$|\langle u_0 - \mathbf{a}, \tilde{u}_t\rangle| + |\langle v_0 - \mathbf{b}, \tilde{u}_t\rangle| + |\langle u_0 - \mathbf{a}, \tilde{v}_t\rangle| + |\langle v_0 - \mathbf{b}, \tilde{v}_t\rangle| \to 0 \tag{4.8}$$

in $\nu \otimes \mathbf{P}^{(\tilde{u}_0,\tilde{v}_0)}$-probability as $t \to \infty$.

As in Section 2, a truncation argument can be used to prove

Corollary 4.1 *If $p(i, j) = p(0, j - i)$, and $\nu = \mathcal{L}[(u_0, v_0)]$ is translation invariant, shift ergodic and satisfies $\int u(0)\, d\nu((u, v)) = a$ and $\int v(0)\, d\nu((u, v)) = b$, then* (4.8) *holds.*

Proof of Proposition 4.1. Let $\mathbf{P} = \nu \otimes \mathbf{P}^{(\tilde{u}_0,\tilde{v}_0)}$. We will show that for all $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathbf{P}[|\langle u - \mathbf{a}, \tilde{u}_t\rangle| > \varepsilon] = 0, \tag{4.9}$$

and it will be clear that our proof will apply to the other terms in (4.8). By Theorem 2.2 of Dawson and Perkins (1998),

$$\tilde{u}_t(i) = P_t \tilde{u}_0(i) + \sum_j \int_0^t p_{t-s}(i, j) dM_s(j),$$

where the series converges in L^2, and the $\{(M_t(i))_{t\geq 0},\ i \in S\}$ are orthogonal square-integrable continuous martingales with square variation functions

$$\langle M(i)\rangle_t = \int_0^t (\tilde{u}_s(i)\tilde{v}_s(i))\, ds.$$

The same result shows that $\langle \tilde{u}_t, \mathbf{1}\rangle$ is a continuous square-integrable martingale. Therefore, (3.1) and (3.2) hold. The orthogonality of the $\{M(i)\}$ and square integrability of the total mass martingale imply (3.5) and (3.15), and the continuity of $\langle \tilde{u}_t, \mathbf{1}\rangle$ implies that (3.16) holds trivially. Consequently, we may apply Theorem 3.2 with the measure ν of the result given by $\mathcal{L}[u_0 - \mathbf{a}]$. $\square$

As with the case of the parabolic Anderson model, we can use duality and Proposition 4.1 to prove a convergence result for (u_t, v_t). Let d be a complete metric inducing the topology of weak convergence on the space $M_1(M^2_{\text{tem}})$ of probability measures on M^2_{tem}.

Theorem 4.1 *Let* $\nu = \mathcal{L}[(u_0, v_0)]$. *Assume either that* (i) $\nu \in \mathcal{M}_{(a,b)}$, *or* (ii) $p(i,j) = p(0, j-i)$ *and* ν *is translation invariant, shift ergodic and satisfies* $\int u(0)\, d\nu((u,v)) = a$ *and* $\int v(0)\, d\nu((u,v)) = b$. *Then*

(a) $d\big(\mathbf{P}^{(u,v)}[(u_t, v_t) \in \cdot],\, \nu_{(a,b)}\big) \to 0$ *in* $d\nu(u,v)$*-probability as* $t \to \infty$.

(b) $\mathcal{L}^{\nu}[(u_t, v_t)] \Rightarrow \nu_{(a,b)}$ *as* $t \to \infty$.

Proof. Clearly (b) follows from (a) by integrating out (u,v) with respect to ν.

For (a), choose $\tilde{u}_0, \tilde{v}_0 \in X_F$ and let $\mathbf{P} = \nu \otimes \mathbf{P}^{(\tilde{u}_0, \tilde{v}_0)}$. Note that under $\mathbf{P}$, (u,v) is a random variable with distribution ν. If we let

$$f(u, v, \phi, \psi) = e^{-\langle u+v-(\mathbf{a}+\mathbf{b}), \phi+\psi\rangle + \mathbf{i}\langle \mathbf{u}-\mathbf{v}-(\mathbf{a}-\mathbf{b}), \phi-\psi\rangle},$$

then by (4.8), $f(u, v, \tilde{u}_t, \tilde{v}_t) \to 1$ in $\mathbf{P}$-probability as $t \to \infty$. By (4.2),

$$\begin{aligned}
&\mathbf{E}^{(u,v)}\big[e^{-\langle u_t+v_t, \tilde{u}_0+\tilde{v}_0\rangle + i\langle u_t - v_t, \tilde{u}_0 - \tilde{v}_0\rangle}\big] \\
&\quad = \mathbf{E}^{(\tilde{u}_0, \tilde{v}_0)}\big[e^{-\langle (u+v), \tilde{u}_t+\tilde{v}_t\rangle + i\langle (u-v), \tilde{u}_t - \tilde{v}_t\rangle}\big] \\
&\quad = \mathbf{E}^{(\tilde{u}_0, \tilde{v}_0)}\big[f(u, v, \tilde{u}_t, \tilde{v}_t) e^{-(a+b)\langle \mathbf{1}, \tilde{u}_t+\tilde{v}_t\rangle + i(a-b)\langle \mathbf{1}, \tilde{u}_t - \tilde{v}_t\rangle}\big].
\end{aligned}$$

By (4.4) and the above,

$$\begin{aligned}
&\mathbf{E}^{(u,v)}\big[e^{-\langle u_t+v_t, \tilde{u}_0+\tilde{v}_0\rangle + i\langle u_t - v_t, \tilde{u}_0 - \tilde{v}_0\rangle}\big] \\
&\quad \to \int e^{-\langle u'+v', \tilde{u}_0+\tilde{v}_0\rangle + i\langle u'-v', \tilde{u}_0 - \tilde{v}_0\rangle}\, d\nu_{(a,b)}(u', v')
\end{aligned} \tag{4.10}$$

in $d\nu(u,v)$-probability as $t \to \infty$. If $\lambda > 0$, then Theorems 2.2(b)(iii) and 1.4 of Dawson and Perkins (1998) show that

$$\begin{aligned}
&\int \Big| \mathbf{E}^{(u,v)}[\langle u_t \pm v_t, \phi_{-\lambda}\rangle] - \int \langle u' \pm v', \phi_{-\lambda}\rangle\, d\nu_{(a,b)}(u', v')\Big|\, d\nu(u,v) \\
&\quad = \int \Big| \langle P_t(u \pm v), \phi_{-\lambda}\rangle - \langle a \pm b, \phi_{-\lambda}\rangle \Big|\, d\nu(u,v) \\
&\quad \le \sum_k \phi_{-\lambda}(k) \int [|P_t u(k) - a| + |P_t v_0(k) - b|]\, d\nu(u,v) \\
&\quad \to 0 \text{ as } t \to \infty,
\end{aligned} \tag{4.11}$$

where we have used (4.5), (4.6), and dominated convergence in the last line. Let X^q_F be the set of $\tilde{u}_0$ in X_F taking on rational values. Choose a sequence $t_n \to \infty$. By (4.10) and (4.11) we may choose a subsequence t_{n_k} such that for ν-a.a. (u,v),

$$\begin{aligned}
&\lim_{k\to\infty} \mathbf{E}^{(u,v)}\big[e^{-\langle u_{t_{n_k}}+v_{t_{n_k}}, \tilde{u}_0+\tilde{v}_0\rangle + i\langle u_{t_{n_k}} - v_{t_{n_k}}, \tilde{u}_0 - \tilde{v}_0\rangle}\big] \\
&\quad = \int e^{-\langle u'+v', \tilde{u}_0+\tilde{v}_0\rangle + i\langle u'-v', \tilde{u}_0 - \tilde{v}_0\rangle}\, d\nu_{(a,b)}(u', v') \text{ for all } \tilde{u}_0, \tilde{v}_0 \in X^q_F,
\end{aligned} \tag{4.12}$$

and

$$\lim_{k\to\infty} \mathbf{E}^{(u,v)}[\langle u_{t_{n_k}} + v_{t_{n_k}}, \phi_{-\lambda}\rangle] = \int \langle u' + v', \phi_{-\lambda}\rangle d\nu_{(a,b)}(u', v') \text{ for all } \lambda \in \mathbb{Q}, \lambda > 0.$$

The latter implies that for ν-a.a. (u, v),

$$\sup_k \mathbf{E}^{(u,v)}[\langle u_{t_{n_k}} + v_{t_{n_k}}, \phi_{-\lambda}\rangle] < \infty \text{ for all } \lambda > 0. \tag{4.13}$$

Fix (u, v) outside of a ν-null set so that (4.12) and (4.13) both hold. A simple approximation argument using (4.13) to bound $\sup_k \mathbf{E}^{(u,v)}[\langle u_{t_{n_k}} + v_{t_{n_k}}, \mathbf{1}_F\rangle]$ for each finite set F, allows one to extend (4.12) to all $\tilde{u}_0, \tilde{v}_0$ in X_F. This extension, together with (4.13), allows us to apply Lemma 2.3(c) of Dawson and Perkins (1998) to conclude that

$$d(\mathbf{P}^{(u,v)}((u_{t_{n_k}}, v_{t_{n_k}}) \in \cdot), \nu_{(a,b)}) \to 0 \text{ as } k \to \infty \;\; \nu - a.a. \; (u, v). \tag{4.14}$$

We have shown that every sequence $t_n \to \infty$ has a subsequence satisfying (4.14), and so (a) is now immediate. $\square$

We include (a) because it is used in Cox and Klenke (1998) to show that, under the appropriate recurrence hyptheses mentioned above, as $t \to \infty$, the "predominant type" near 0 changes infinitely often. (Recall in this setting that there is extinction of one type in the equilibrium limit.) This result was the original motivation for this work.

These methods also apply to the continuous version of (4.1). Let $\dot{W}_1$ and $\dot{W}_2$ be independent space-time white noises on $\mathbb{R}_+ \times \mathbb{R}_+$ and consider the system of stochastic partial differential equations

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x) + (cu(t,x)v(t,x))^{1/2}\dot{W}_1(t,x),\\ \frac{\partial v}{\partial t}(t,x) &= \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x) + (cu(t,x)v(t,x))^{1/2}\dot{W}_2(t,x).\end{aligned} \tag{4.15}$$

See Dawson and Perkins (1998) for a precise interpretation of this pair of equations. Let $|f|_\lambda = \sup\{f(x)e^{\lambda|x|} : x \in \mathbb{R}\}$ and assume that $u_0, v_0 \in C^+_{\text{tem}}$, where

$$C^+_{\text{tem}} = \{f : \mathbb{R} \to [0, \infty) : \; f \text{ is continuous, and } |f|_\lambda < \infty \text{ for all } \lambda < 0\}.$$

We topologize C^+_{tem} so that $f_n \to f$ in C^+_{tem} if and only if $|f_n - f|_\lambda \to 0$ for all $\lambda < 0$.

By Theorem 1.7 of Dawson and Perkins (1998), there is a unique (in law) solution (u_t, v_t) of (4.15) satisfying $(u_\cdot, v_\cdot) \in C([0,\infty), (C^+_{\text{tem}})^2)$. Uniqueness was first shown by Mytnik (1998) through the continuous analogue of (4.2). Let $\mathbf{P}^{(u,v)}$ denote the law of the solution of (4.15) with $(u_0, v_0) = (u, v)$, and let $\mathbf{P}^\nu$ denote this law if $\mathcal{L}[u_0, v_0] = \nu$, a probability measure on $(C^+_{\text{tem}})^2$.

As in Dawson and Perkins (1998), for the purpose of convergence to equilibrium, we weaken the topology on C^+_{tem}. Let $M^c_{\text{tem}} = C^+_{\text{tem}}$ but the weak topology given by: $f_n \to f$ in M^c_{tem} if and only if $\lim \int f_n(x)\phi(x)\,dx = \int f(x)\phi(x)\,dx$ for all continuous ϕ satisfying $|\phi|_\lambda < \infty$ for some $\lambda > 0$. As before, let $\mathbf{P}^{a,b}$ denote the law of a planar Brownian motion started at $(a, b) \in \mathbb{R}^2_+$, and let T be the Brownain motion's first exit time from the first quadrant. Let $\mathbf{a}, \mathbf{b}$ denote the constant functions on $\mathbb{R}$. Theorem 1.8 of Dawson and Perkins (1998) states that, for $a, b > 0$,

$$\mathbf{P}^{\mathbf{a},\mathbf{b}}[(u_t, v_t) \in \cdot] \Rightarrow \nu^c_{(a,b)} = \mathbf{P}^{a,b}[(\mathbf{B}^1_T, \mathbf{B}^2_T) \in \cdot] \tag{4.16}$$

in $(M^c_{\text{tem}})^2$ as $t \to \infty$.

For $a, b \geq 0$ let $\mathcal{M}_{(a,b)}$ be the set of probability measures ν on $(C^+_{\text{tem}})^2$ such that

$$\sup_x \int (u^2(x) + v^2(x))\, d\nu(u, v) < \infty, \tag{4.17}$$

$$\lim_{t\to\infty} \int [(P_t u(x) - a)^2 + (P_t v(x) - b)^2]\, d\nu(u, v) = 0, \quad \text{all } x \in \mathbb{R}. \tag{4.18}$$

where now P_t is the semigroup of one-dimensional Brownian motion.

Although we may no longer invoke the "discrete" Theorem 3.2, it is easy to argue directly as in Section 2, using (4.16), the continuous analogue of (4.2), and Theorem 6.1 in Dawson and Perkins, to prove the analogue of Theorem 4.1 given below. Let d be a metric on $M_1((M^c_{tem})^2)$ inducing the topology of weak convergence on this space of probability laws.

Theorem 4.2 *Assume $a, b \geq 0$, and $\nu = \mathcal{L}[(u_0, v_0)] \in \mathcal{M}^c_{a,b}$.*

(a) *$d\big(\mathbf{P}^{(u,v)}[(u_t, v_t) \in \cdot],\, \nu^c_{(a,b)}\big) \to 0$ in $d\nu(u, v)$-probability as $t \to \infty$.*

(b) *$\mathcal{L}[(u_t, v_t)] \Rightarrow \nu^c_{(a,b)}$ as $t \to \infty$ in the sense of weak convergence of probability measures on $(M^c_{tem})^2$.*

5 Linear systems with values in $[0, \infty)^{\mathbb{Z}^d}$

We consider a subclass of the linear systems treated in Chapter IX of Liggett (1985). We set $S = \mathbb{Z}^d$, and, following Liggett (1985), use x, i, j, k, l to denote generic elements of $\mathbb{Z}^d$. Our process will be denoted η_t, and takes values in $[0, \infty)^S$. Let $A(x; i, j)$, $x, i, j \in \mathbb{Z}^d$, be nonnegative random variables. It is convenient to view $A(x) = A(x; i, j)$, $i, j \in \mathbb{Z}^d$, as a random matrix indexed by $\mathbb{Z}^d \times \mathbb{Z}^d$. Let $\mathfrak{M}$ denote the set of such infinite matrices. Given a configuration $\eta \in [0, \infty)^S$, let $A(x)\eta$ be the configuration defined by $A(x)\eta(i) = \sum_j A(x; i, j)\eta(j)$. The process η_t is defined as follows. At each $x \in \mathbb{Z}^d$ there is a rate one exponential alarm clock. If the clock at site x goes off at time t, the configuration $\eta_{t-}$is replaced by $A(x)\eta_{t-}$. At each such time and site, independent instances of the $A(x)$ are used and they are identically distributed in time for each site x. The smoothing and potlatch processes introduced in Spitzer (1981) are the main examples of this type of process we will consider.

Let μ be the infinite measure on $V = \mathbb{Z}^d \times \mathfrak{M}$ given by

$$\mu(C \times D) = \sum_{x\in C} \mathbf{P}[A(x) \in D], \qquad C \subset \mathbb{Z}^d,\ \text{Borel } D \subset \mathfrak{M}. \tag{5.1}$$

We assume there is a finite constant M such that

$$\sup_i \int \Big[\sum_j |A(x; i, j) - \delta(i, j)|\Big] + \Big[\sum_j |A(x; i, j) - \delta(i, j)|\Big]^2 d\mu(x, A) \leq M. \tag{5.2}$$

This condition was introduced in Chapter IX of Liggett (1985) to ensure existence of the desired process, and finiteness of second moments of its coordinates (see (1.4), Lemma 1.6 and (3.2) of that reference). By Theorem IX.1.6 of Liggett (1985), there is a strictly positive, summable function α on $\mathbb{Z}^d$ such that

$$\mathbf{E}\Big[\sum_{x\in\mathbb{Z}^d}\sum_{i:\, i\neq j}\alpha(i)A(x;i,j)\Big]\le M\alpha(j). \tag{5.3}$$

Let $X = X_\alpha = \{\eta \in [0,\infty)^{\mathbb{Z}^d} : \langle \eta,\alpha\rangle < \infty\}$, endowed with the topology of pointwise convergence. Under (5.2) there is a well defined Markov process η_t taking values in X which is specified by the description above (see Theorem IX.1.14 of Liggett (1985) for a precise statement).

Let $\tilde{A}(x)$ denote the transpose of $A(x)$. Then we may define another process $\tilde{\eta}_t$ using the $\tilde{A}(x)$ instead of the $A(x)$. In order that $\tilde{\eta}_t$ be well defined, we assume that (5.2) holds with A replaced by $\tilde{A}$. As shown in Liggett (1985), η_t and $\tilde{\eta}_t$ are dual processes, exactly as in the parabolic Anderson model. Given $\eta_0 \in X$ and $\tilde{\eta}_0 \in X_F$,

$$\langle \eta_t,\tilde{\eta}_0\rangle \stackrel{\mathrm{d}}{=} \langle \eta_0,\tilde{\eta}_t\rangle. \tag{5.4}$$

Now define $\tilde{\mu}$ as μ (5.1), but with $\tilde{A}$ in place of A, and also

$$\gamma_x = \mathbf{E}[(A-I)(x)], \qquad \tilde{\gamma}_x = \mathbf{E}[(\tilde{A}-I)(x)], \tag{5.5}$$

and

$$\gamma = \sum_x \gamma_x = \int (A-I)d\mu(x,A), \qquad \tilde{\gamma} = \sum_x \tilde{\gamma}_x = \int (\tilde{A}-I)d\mu(x,A). \tag{5.6}$$

Since the $A(x)$ are nonnegative, $\gamma(i,j)$ and $\tilde{\gamma}(i,j)$ are nonnegative for $i \neq j$. Clearly (5.2) implies all these coefficients are finite and $\sup_i \sum_j |\gamma(i,j)| + |\tilde{\gamma}(i,j)| < \infty$. We also assume that

$$\gamma\mathbf{1} = \mathbf{1}\gamma = 0. \tag{5.7}$$

Thus, γ and $\tilde{\gamma}$ are q–matrices for rate-one continuous time Markov chains on S whose transition semigroups are γ_t and $\tilde{\gamma}_t$, respectively. (Note that, for simplicity, we have omitted the quantities $a(i,j)$ in Liggett (1985).)

We now construct η_t as the unique solution of a stochastic differential equation driven by a Poisson point process. Although this is not essential for our arguments, it provides some additional methodology for the study of these linear systems, and does not quite follow from the general constructions in Kurtz and Protter (1996). We do this for $\eta_0 \in X_f = \{\eta \in [0,\infty)^{\mathbb{Z}^d} : \langle \eta,\mathbf{1}\rangle < \infty\}$, although our construction holds more generally. Give X_f the topology of weak convergence of finite measures on $\mathbb{Z}^d$ and let $D(X_f)$ denote the Skorokhod space of right continuous X_f-valued paths on $[0,\infty)$ with left limits. Let N be a Poisson point process on $[0,\infty)\times V$ with intensity $m\times\mu$, with respect to the filtration $\mathcal{F}_t$ on $(\Omega,\mathcal{F},\mathbf{P})$ (m is Lebesgue measure). Let $\hat{N}$ denote the orthogonal martingale measure

$$\hat{N}(t,U) = N([0,t],U) - t\mu(U) \text{ for } \mu(U) < \infty. \tag{5.8}$$

Recall that $\int_0^t \int f(s,\omega,x,A)\hat{N}(ds,dx,dA)$ is defined and is a local martingale for a large class of integrands f (see section II.3 of Ikeda and Watanabe (1981)). For example if $f \in \mathcal{L}^2$, the class of $\mathcal{F}_t$-predictable$\times$ Borel measurable functions such that $\mathbf{E}[\int_0^t \int_V f(s,\omega,x,A)^2 d\mu ds] < \infty$ for all $t > 0$, then $\int_0^t \int f(s,\omega,x,A)\hat{N}(ds,dx,dA)$ is a square-integrable martingale with square function $\int_0^t \int_V f(s,\omega,x,A)^2 d\mu\, ds$. For square summable $\eta \in [0,\infty)^S$, let

$$q(\eta;i,j) = \int_V \sum_{i'} (A(x;i,i') - \delta(i,i'))\eta(i') \sum_{j'} (A(x;j,j') - \delta(j,j'))\eta(j')\, d\mu\,. \tag{5.9}$$

An application of Hölder's inequality and (5.2) implies $|q(\eta;i,j)| \le M\|\eta\|_2^2 < \infty$.

Proposition 5.1 (a) *There is a unique process $\{\eta_t : t \ge 0\}$ with paths in $D(X_f)$ such that*

$$\eta_t = \eta_0 + \int_0^t \int_V (A-I)\eta_{s-} N(ds,dx,dA) \quad \textit{for all } t \ge 0, \quad \textit{a.s.} \tag{5.10}$$

Its law coincides with the law of the process constructed in Theorem IX.1.14 of Liggett (1985) (and described above).

(b) *The total mass process*

$$\langle \eta_t, \mathbf{1}\rangle = \langle \eta_0, \mathbf{1}\rangle + \int_0^t \int_V \langle (A-I)\eta_{s-}, \mathbf{1}\rangle \hat{N}(ds,dx,dA) \tag{5.11}$$

is a non-negative square-integrable martingale with predictable square function

$$C_t = \int_0^t \int_V \langle (A-I)\eta_{s-}, \mathbf{1}\rangle^2\, d\mu\, ds = \int_0^t \langle \mathbf{1}, q(\eta_{s-})\mathbf{1}\rangle ds \in L^1. \tag{5.12}$$

Moreover if we let

$$\bar{C}_t = \int_0^t \int_V \langle |(A-I)|\eta_{s-}, \mathbf{1}\rangle^2\, d\mu\, ds, \tag{5.13}$$

then $\bar{C}_t$ is also integrable.

(c) *We have*

$$\eta_t = \eta_0 + \int_0^t \gamma \eta_s\, ds + M_t, \tag{5.14}$$

where for each $i \in \mathbb{Z}^d$,

$$M_t(i) = \int_0^t \int_V (A-I)\eta_{s-}(i)\hat{N}(ds,dx,dA) \tag{5.15}$$

is a square-integrable martingale satisfying

$$\langle M(i), M(j)\rangle_t = \int_0^t q(\eta_{s-};i,j)\, ds. \tag{5.16}$$

Proof. (a) Let $\{S_n\}$ be a sequence of finite sets increasing to $\mathbb{Z}^d$. As in Liggett (1985) we start with solutions η_t^n of the truncated system

$$\eta_t^n = \eta_0 + \int_0^t \int_V \mathbf{1}_{S_n}(x)(A-I)\eta_{s-}^n N(ds,dx,dA). \tag{5.17}$$

Such solutions exist and are unique because N has finite intensity on compact time intervals if x is restricted to S_n. With our finite initial conditions we may argue as in Theorem IX.1.14 of Liggett (1985) but with $\alpha \equiv 1$ to conclude that the limits

$$\lim_{n\to\infty} \mathbf{E}[\eta_t^n] = m_t, \quad \lim_{n\to\infty} \mathbf{E}[\langle \eta_t^n, \mathbf{1}\rangle] = \langle m_t, \mathbf{1}\rangle, \tag{5.18}$$

exist, and

$$\mathbf{E}[\langle \eta_t^n, \mathbf{1}\rangle] \le \langle \eta_0, \mathbf{1}\rangle\, e^{3Mt} \text{ for all } n \in \mathbb{N}, t > 0. \tag{5.19}$$

Define the matrices $a_x = \mathbf{E}[|(A-I)(x)|]$ and

$$a = \sum_x a_x = \int_V |(A-I)|\, d\mu. \tag{5.20}$$

For $m \le n$,

$$\begin{aligned}
&\mathbf{E}\Big[\sup_{s\le t}\langle |\eta_s^n - \eta_s^m|, 1\rangle\Big] \\
&\quad\le \mathbf{E}\left[\int_0^t \int_V \mathbf{1}_{S_m}(x)\langle |(A-I)||\eta_{s-}^n - \eta_{s-}^m|, \mathbf{1}\rangle\, N(ds,dx,dA)\right] \\
&\qquad + \mathbf{E}\left[\int_0^t \int_V \mathbf{1}_{S_n - S_m}(x)\langle |(A-I)|\eta_{s-}^n, \mathbf{1}\rangle\, N(ds,dx,dA)\right] \\
&\quad= \int_0^t \sum_{x\in S_m} \langle a_x \mathbf{E}[|\eta_s^n - \eta_s^m|], \mathbf{1}\rangle\, ds + \int_0^t \sum_{x \in S_n - S_m} \langle a_x \mathbf{E}[\eta_s^n], \mathbf{1}\rangle\, ds \\
&\quad\le \left[\sup_j a\mathbf{1}(j)\right]\left[\int_0^t \mathbf{E}\left[\langle |\eta_s^n - \eta_s^m|, \mathbf{1}\rangle\right]\, ds + \int_0^t \langle |\mathbf{E}[\eta_s^n] - m_s|, \mathbf{1}\rangle\, ds\right] \\
&\qquad + \int_0^t \sum_{x\in S_n - S_m} \langle a_x m_s, \mathbf{1}\rangle\, ds.
\end{aligned} \tag{5.21}$$

Denote the last term in the above by $\varepsilon_{m,n}(t)$. Note that (5.2) (applied to $\tilde{A}$) implies $\sup_j \sum_i a(i,j) \le M$. This together with (5.19) implies that

$$\int_0^t \langle a m_s, \mathbf{1}\rangle\, ds \le M \int_0^t \langle \eta_0, \mathbf{1}\rangle e^{3Ms}\, ds, \tag{5.22}$$

and so $\sup_{t\le T}\varepsilon_{m,n}(t) \to 0$ as $m,n\to\infty$ by dominated convergence. Clearly (5.18) implies that for each s, $\lim_{n\to\infty}\langle |\mathbf{E}[\eta_s^n] - m_s|, \mathbf{1}\rangle = 0$, and so (5.19) and dominated convergence show that the "middle term" on the right side of (5.21) approaches 0 as m,n approach ∞. Substituting the above bounds into the right side of (5.21), we arrive at

$$\mathbf{E}\left[\sup_{s\le t}\langle |\eta_s^n - \eta_s^m|, \mathbf{1}\rangle\right] \le M\int_0^t \mathbf{E}\left[\langle |\eta_s^n - \eta_s^m|, \mathbf{1}\rangle\right]\, ds + \delta_{m,n}(t), \tag{5.23}$$

where $\delta_{m,n}$ is an increasing function in t which approaches 0 as $m,n\to\infty$. This shows that for each $T>0$,

$$\mathbf{E}\left[\sup_{t\le T}\langle |\eta_t^n - \eta_t^m|, \mathbf{1}\rangle\right] \le \delta_{m,n}(T)e^{MT} \to 0 \text{ as } m,n\to\infty. \tag{5.24}$$

By taking a subsequence we may assume that there is a process η_t with sample paths in $D(X_f)$ such that $\lim_{n\to\infty}\sup_{t\le T}\langle|\eta^n_t-\eta_t|,\mathbf{1}\rangle\to 0$ for all $T>0$ a.s. The above argument and Fatou's lemma also show that

$$\mathbf{E}\Big[\sup_{t\le T}\Big|\int_0^t\int\Big(\mathbf{1}_{S_m}(x)\langle(A-I)(\eta^m_{s-}-\eta_{s-}),\mathbf{1}\rangle\Big)N(ds,dx,dA)\Big|\Big]\to 0 \text{ as } m\to\infty. \tag{5.25}$$

Now let $n\to\infty$ in (5.17) to derive (5.10). The fact that the law of this solution coincides with that of the process constructed in Theorem IX.1.14 of Liggett (1985) is immediate from the construction of the latter as the weak limit of solutions to (5.17).

Turning to the uniqueness of solutions to (5.10), we let η denote any solution of this equation. If $T_n=\inf\{t:\langle\eta_t,\mathbf{1}\rangle\ge n\}$, then

$$\begin{aligned}\mathbf{E}[\langle\eta_{t\wedge T_n},\mathbf{1}\rangle]&\le\langle\eta_0,\mathbf{1}\rangle+\mathbf{E}\left[\int_0^{t\wedge T_n}\int_V\langle|(A-I)|\eta_{s-},\mathbf{1}\rangle d\mu ds\right]\\&\le\langle\eta_0,\mathbf{1}\rangle+M\mathbf{E}\left[\int_0^{t\wedge T_n}\langle\eta_{s-},\mathbf{1}\rangle\,ds\right]\end{aligned}\tag{5.26}$$

again by applying (5.2) to $\tilde{A}$. The right side of the above is clearly finite, and so $f_n(t)=\mathbf{E}[\langle\eta_{t\wedge T_n},\mathbf{1}\rangle]$ is a finite function satisfying

$$f_n(t)\le\langle\eta_0,\mathbf{1}\rangle+M\int_0^t f_n(s)\,ds. \tag{5.27}$$

From this and Fatou's lemma, we finally arrive at the bound

$$\mathbf{E}[\langle\eta_t,\mathbf{1}\rangle]\le\langle\eta_0,\mathbf{1}\rangle e^{Mt}. \tag{5.28}$$

The uniqueness now follows as in Section 9.1 of Kurtz and Protter (1996). We apply their reasoning with $F_i(\eta_{s-},A,x)=(A-I)\eta_{s-}(i)$, and note that (9.2) of that reference holds with their $a_{i,j}$ equal to our $a(i,j)$, and their (9.6) holds with $p=1$, $q=\infty$, and $\alpha\equiv 1$ by (5.2). Of course, F_i is not bounded as in that reference, but this was only used to derive (5.28) and so is not needed.

Parts (b) and (c) of Theorem IX.2.2 of Liggett (1985) shows $\langle\eta_t,\mathbf{1}\rangle$ is a martingale (this is also immediate from the representation and integrability conditions derived below). Note that

$$\begin{aligned}\mathbf{E}[\bar{C}_{t\wedge T_n}]&=\mathbf{E}\left[\int_0^{t\wedge T_n}\int_V\langle|(A-I)|\eta_{s-},\mathbf{1}\rangle^2\,d\mu\,ds\right]\\&\le\mathbf{E}\left[\int_0^{t\wedge T_n}\langle\eta_s,\mathbf{1}\rangle^2 2\sup_i\int_V(\mathbf{1}|(A-I)|(i))^2 d\mu\,ds\right]\\&\le\mathbf{E}\left[\int_0^{t\wedge T_n}\langle\eta_s,\mathbf{1}\rangle^2 2M\,ds\right]<\infty,\end{aligned}\tag{5.29}$$

where we again use (5.2) for $\tilde{A}$. The square integrability of $\langle\eta_{t\wedge T_n},\mathbf{1}\rangle$ is now clear from the above and (5.10). We also see from the above that

$$\begin{aligned}\mathbf{E}\left[\langle\eta_{t\wedge T_n},\mathbf{1}\rangle^2\right]&\le 2\langle\eta_0,\mathbf{1}\rangle^2+2\mathbf{E}[\bar{C}_{t\wedge T_n}]\\&\le 2\langle\eta_0,\mathbf{1}\rangle^2+4M\mathbf{E}\left[\int_0^t\langle\eta_{s\wedge T_n},\mathbf{1}\rangle^2\,ds\right].\end{aligned}\tag{5.30}$$

It follows that

$$\mathbf{E}\left[\langle\eta_{t\wedge T_n},\mathbf{1}\rangle^2\right]\le 2\langle\eta_0,\mathbf{1}\rangle^2 e^{4Mt}, \tag{5.31}$$

and Fatou's lemma shows that $\langle\eta_t,\mathbf{1}\rangle$ is square-integrable. Use this and let $n\to\infty$ in (5.29) to obtain the integrability of $\bar{C}(t)$ for each t.

The above integrability allows us to rewrite (5.10) as (see Section II.3 of Ikeda and Watanabe (1981))

$$\eta_t=\eta_0+\int_0^t\int_V (A-I)\eta_{s-}\,d\mu\,ds+\int_0^t\int_V (A-I)\eta_{s-}\hat{N}(ds,dx,dA), \tag{5.32}$$

and this gives the expression in (c). The formulae for the predictable square functions are immediate from our earlier discussion on the $\hat{N}$ stochastic integrals. To see the representation for $\langle\eta_t,\mathbf{1}\rangle$ in (b) simply sum (5.10) over the sites in S. The stochastic integrals converge in L^1 by (5.2) and integrability of the total mass, and (5.7) shows that the resulting integral with respect to N equals the same integral with respect to the martingale measure $\hat{N}$. The integrability of $\bar{C}(t)$, and hence of $C(t)$, implies that $C(t)$ is the predictable square function of this stochastic integral representation of the total mass process (for it shows the integrand is in $\mathcal{L}^2$). $\square$

Remark 5.1 *Note that*

$$\begin{aligned}\sum_{i,j}&|\langle M(i),M(j)\rangle|_t\\ &\le\int\sum_{i,j,i',j'}\int_V |A(i,i')-\delta(i,i')||A(j,j')-\delta(j,j')|\,d\mu\,\eta_s(i')\eta_s(j')\,ds\\ &=\bar{C}_t\end{aligned} \tag{5.33}$$

and so the above integrability of $\bar{C}_t$ *implies* (3.5) *in Section 3.*

If $\tilde{\eta}_0\in X_F$, the above shows that $\langle\tilde{\eta}_t,\mathbf{1}\rangle$ is a nonnegative martingale, and hence converges almost surely. In view of the duality relation (5.4), this implies weak convergence of η_t starting from constant initial states. That is, if $\theta\in[0,\infty)$ and η_t^θ is the linear system with initial state $\eta_0^\theta=\boldsymbol{\theta}$, then there is a probability measure ν_θ on X such that

$$\mathcal{L}[\eta_t^\theta]\Rightarrow\nu_\theta\qquad\text{as } t\to\infty$$

(as probability measures on $[0,\infty)^{\mathbb{Z}^d}$ with the product topology). Moreover, if $\phi\in X_F$,

$$\begin{aligned}\mathbf{E}^\phi[e^{-\theta\langle\mathbf{1},\tilde{\eta}_t\rangle}]&=\mathbf{E}[e^{-\langle\eta_t^\theta,\phi\rangle}]\\ &\to\int e^{-\langle\eta,\phi\rangle}\,d\nu_\theta(\eta)\qquad\text{as } t\to\infty.\end{aligned} \tag{5.34}$$

For $\theta\in[0,\infty)$, define $\mathcal{M}_\theta$ to be the collection of probability measures ν on X such that (recall that γ_t is the semigroup of the q–matrix γ defined in (5.6))

$$\sup_k\int\eta^2(k)\,d\nu(\eta)<\infty, \tag{5.35}$$

$$\lim_{t\to\infty}\int(\gamma_t\eta(k)-\theta)^2\,d\nu(\eta)=0,\quad k\in\mathbb{Z}^d. \tag{5.36}$$

Define $\tilde{C}$ and $\bar{\tilde{C}}$ as in Proposition 5.1, but with $(\tilde{\eta}, \tilde{A})$ in place of (η, A). Hence $\tilde{C}_t = \int_0^t \langle \mathbf{1}, \tilde{q}(\tilde{\eta}_s)\mathbf{1}\rangle ds$ is the predictable square function of the non-negative martingale $\langle \tilde{\eta}, \mathbf{1}\rangle$. For the following result, we recall that our standing hypotheses (5.2) and (5.7) are in effect, and that q is the covariation kernel defined in (5.9).

Proposition 5.2 *Let $\theta \in [0, \infty)$ and $\nu = \mathcal{L}[\eta_0] \in \mathcal{M}_\theta$. Assume that $\tilde{\eta}_0 \in X_f$, and*

$$\int_0^\infty \langle \mathbf{1}, |q(\tilde{\eta}_t)|\mathbf{1}\rangle \, dt < \infty \quad a.s.\ \mathbf{P}^{\tilde{\eta}_0}. \tag{5.37}$$

Then

$$\langle \eta_0 - \boldsymbol{\theta}, \tilde{\eta}_t\rangle \to 0 \tag{5.38}$$

in $\nu \otimes \mathbf{P}^{\tilde{\eta}_0}$-probability as $t \to \infty$.

As in Section 2, the second moment condition can be weakened in the *random walk* setting

$$\gamma(i, j) = \gamma(0, j - i), \qquad i, j \in \mathbb{Z}^d. \tag{5.39}$$

A truncation argument can be used to prove

Corollary 5.1 *If* (5.37) *holds, γ is the q-matrix of random walk, and $\nu = \mathcal{L}[\eta_0]$ is translation invariant, shift ergodic and satisfies $\int \eta(0)\, d\nu(\eta) = \theta$, then* (5.38) *holds.*

We may apply the duality argument used in the previous sections to obtain the following result.

Theorem 5.1 *Assume* (5.37) *holds for all $\tilde{\eta}_0 \in X_f$, and either* (i) $\mathcal{L}[\eta_0] \in \mathcal{M}_\theta$, *or* (ii) *$\gamma$ is the q-matrix of a random walk and $\mathcal{L}[\eta_0]$ is translation invariant, shift ergodic and satisfies $\int \eta(0)\, d\nu(\eta) = \theta$. Then $\mathcal{L}[\eta_t] \Rightarrow \nu_\theta$ as $t \to \infty$.*

Remark 5.2 *It is possible to formulate and prove a stronger result, analogous to part (a) of Theorem 4.1.*

Remark 5.3 *Theorems 3.17 and 3.29 in Chapter IX of Liggett (1985) give conditions under which $\mathcal{L}[\eta_t] \Rightarrow \nu_\theta$ if γ is the q-matrix of a random walk, and $\mathcal{L}[\eta_0]$ is translation invariant and shift ergodic. Theorem 5.1 above solves, at least in part, Problem 5 of Chapter IX in Liggett (1985).*

Proof of Proposition 5.2. Apply Itô's lemma to $\tilde{\gamma}_{t-s}\tilde{\eta}_s$ in the decomposition for $\tilde{\eta}$ in Proposition 5.1(c) to see that

$$\tilde{\eta}_t = \tilde{\gamma}_t \tilde{\eta}_0 + \tilde{N}_t^t, \tag{5.40}$$

where

$$\tilde{N}_s^t(i) = \sum_j \int_0^s \tilde{\gamma}_{t-r}(i, j) d\tilde{M}_r(j) \equiv \int_0^s \tilde{\gamma}_{t-r} d\tilde{M}_r(i), \qquad 0 \le s \le t, \tag{5.41}$$

and $\tilde{M}$ is defined as in Proposition 5.1(b) with $\tilde{A}$, $\tilde{\eta}_s$, in place of A, η_s. Our assumption (5.2) easily gives us enough summability to verify that the bounded variation term in the above vanishes, and Remark 5.1 shows the above series of martingales is L^2-convergent. Therefore (3.2) holds, while (3.1) is immediate from Proposition 5.1, as is (3.5) (see Remark 5.1). Finally, (3.6) is precisely (5.37) (by (5.16)), and so we may apply Theorem 3.2 with ν equal to the law of $\eta_0 - \theta$. $\square$

With a view to verifying (5.37) we would like to infer $\tilde{C}_\infty < \infty$ from the a.s. convergence of $\langle \tilde{\eta}_t, \mathbf{1} \rangle$. To apply Lemma 3.1 we will need

$$\exists\, K > 0 : \sup_j \sum_i [|(\tilde{A} - I)(x)(i,j)| + |(A - I)(x)(i,j)|] \leq K \quad \mu\text{- a.a. } (x, A). \tag{5.42}$$

Proposition 5.3 *Let $\tilde{\eta}_0 \in X_f$, and assume either condition* (5.42)*, or*

$$\sup_t \mathbf{E}^{\tilde{\eta}_0}[\langle \tilde{\eta}_t, \mathbf{1} \rangle^2] < \infty. \tag{5.43}$$

Then

$$\tilde{C}_\infty = \int_0^\infty \langle \mathbf{1}, q(\tilde{\eta}_s) \mathbf{1} \rangle \, ds < \infty \text{ a.s.} \tag{5.44}$$

Proof. For $n \in \mathbb{N}$ we define $T_n = \inf\{t \geq 0 : \langle \tilde{\eta}_t, \mathbf{1} \rangle \geq n\}$. If (5.42) holds and $n \geq \langle \tilde{\eta}_0, \mathbf{1} \rangle$, then Proposition 5.1(b) shows that $|\langle \tilde{\eta}_{T_n} - \tilde{\eta}_{T_n -}, \mathbf{1} \rangle| \leq nK$ a.s. on $\{T_n < \infty\}$, where K is as in (5.42) and so Lemma 3.1 implies the required conclusion. If the other hypothesis holds, then $\mathbf{E}[\tilde{C}_\infty] < \infty$ is obvious. $\square$

Theorem 5.2 *Assume either that condition* (5.42) *holds, or for each $\tilde{\eta}_0 \in X_f$, $\sup_t \mathbf{E}^{\tilde{\eta}_0}[\langle \tilde{\eta}_t, \mathbf{1} \rangle^2] < \infty$. Assume also that $q(\eta; i, j) \geq 0$ for all $i, j \in \mathbb{Z}^d$ and square summable η. Then* (5.37) *holds, and therefore, if $\mathcal{L}[\eta_0] \in \mathcal{M}_\theta$ or $\mathcal{L}[\eta_0]$ and γ are as in Corollary 5.1, then $\mathcal{L}[\eta_t] \Rightarrow \nu_\theta$ as $t \to \infty$.*

Proof. As $q(\tilde{\eta}_t; i, j) \geq 0$, clearly the integral in (5.37) is just $\tilde{C}_\infty$. Proposition 5.3 therefore implies (5.37). The result now follows from Theorem 5.1. $\square$

5.1 The Smoothing and Potlatch Processes

The two primary examples we consider are the *smoothing process* and the *potlatch process*, which are dual to one another. Let $p(i,j)$ be a doubly stochastic Markov chain matrix, and let W be a bounded nonnegative random variable with mean 1. For the smoothing process, when the clock at x goes off, $\eta(x)$ is replaced by $W_x p \eta(x)$, and all other coordinates are left fixed. Here, the $W_x, x \in \mathbb{Z}^d$ are iid with law $\mathcal{L}[W]$. For the potlatch process, when the clock at site x goes off, the configuration η is replaced by $A(x)\eta$, where $A(x)\eta(x) = p(x,x) W_x \eta(x)$, and for $i \neq x$, $A(x)\eta(i) = \eta(i) + W_x \eta(x) p(x,i)$. That is, the value $\eta(x)$ is removed, multiplied by W_x, and then redistributed according to the kernel $p(x,i)$. The two processes are respective duals.

For the smoothing process,

$$A(x; i, j) = \begin{cases} 1, & i = j \neq x, \\ W_x p(i,j), & i = x, \\ 0, & \text{else}, \end{cases} \tag{5.45}$$

and hence

$$(A - I)(x; i, j) = ((W_x p) - I)(i, j) \delta(i, x) \,.$$

It follows that

$$(|(A-I)(x)| + |(\tilde{A}-I)(x)|)\mathbf{1}(i) \leq 2(W_x+1)\delta(i,x)\,. \tag{5.46}$$

The boundedness condition on W shows that (5.42) holds and also gives (5.2) (although square integrability suffices for the latter). Note also that (5.7) is true because W has mean 1. We also have

$$\begin{aligned} q(\eta;i,j) &= \mathbf{E}[(((Wp)-I)\eta)(i)^2]\delta(i,j) \\ &= (\mathbf{E}[W^2-1]p\eta(i)^2 + (p-I)\eta(i)^2)\delta(i,j) \geq 0. \end{aligned} \tag{5.47}$$

Thus, Theorem 5.2 applies when $\tilde{\eta}_t$ is the smoothing process, and hence we obtain a convergence result for the dual of the smoothing process, which is the potlatch process.

Theorem 5.3 *Let η_t be the potlatch process, and assume that $\mathcal{L}[\eta_0] \in \mathcal{M}_\theta$, or that $p(i,j)$ is a random walk kernel and $\mathcal{L}[\eta_0]$ is translation invariant, shift ergodic and satisfies $E[\eta_0(0)] = \theta$. Then $\mathcal{L}[\eta_t] \Rightarrow \nu_\theta$ as $t \to \infty$.*

For the potlatch process,

$$A(x;i,j) = \begin{cases} 1, & i = j \neq x, \\ W_x p(j,i), & j = x, \\ 0, & \text{else}, \end{cases} \tag{5.48}$$

and hence

$$(A-I)(x;i,j) = (W_x p(j,i) - \delta(j,i))\delta(j,x)$$

and we have already checked (5.2) and (5.42) (see (5.46)). As before, (5.7) also holds. Furthermore,

$$q(\eta;i,j) = \sum_k \mathbf{E}[(Wp(k,i)-\delta(k,i))(Wp(k,j)-\delta(k,j))]\eta(k)^2,$$

and

$$\langle \mathbf{1}, q(\eta)\mathbf{1}\rangle = (\mathbf{E}[W^2]-1)\langle \eta^2, \mathbf{1}\rangle.$$

Now, in the case that $\mathbf{P}[W=1] < 1$, $\mathbf{E}[W^2] > 1$, so it follows from Proposition 5.3 that

$$\int_0^\infty \langle \eta_t^2, \mathbf{1}\rangle\, dt < \infty \quad \text{a.s. } \mathbf{P}^{\tilde{\eta}_0}. \tag{5.49}$$

On the other hand, from the above expression for $q(\eta;i,j)$,

$$\langle \mathbf{1}, |q(\eta)|\mathbf{1}\rangle \leq (\mathbf{E}[W^2]+3)\langle \eta^2, \mathbf{1}\rangle. \tag{5.50}$$

Therefore, (5.37) holds by (5.49). Consequently, we obtain a convergence result for the smoothing process via Theorem 5.1 in the case that $\mathbf{P}[W=1] < 1$.

In the case that $\mathbf{P}[W=1] = 1$, then $\langle \eta_t, \mathbf{1}\rangle = \langle \eta_0, \mathbf{1}\rangle$ with probability one for all t, and there is no obvious way to obtain (5.37) in general. We now give a direct derivation of (5.49) that holds in the transient, random walk (i.e., (5.39) holds) case. Let $\hat{p}(i,j)$ be the symmetrized kernel, $\hat{p}(i,j) = (p(i,j)+p(j,i))/2$, and let $\hat{G}(i,j) = \int_0^\infty \hat{p}_s(i,j)\,ds$ (recall (1.1)).

Proposition 5.4 *Let η_t be the potlatch process. Assume* (5.39), $\hat{G}(0,0) < \infty$, *and*

$$\mathbf{E}[W^2] < \frac{\hat{G}(0,0)}{(p\hat{G}\tilde{p})(0,0)}. \tag{5.51}$$

Then for any initial $\eta_0 \in X_f$,

$$\int_0^\infty \mathbf{E}[\langle \eta_t^2, \mathbf{1}\rangle]\, dt < \infty. \tag{5.52}$$

Remark 5.4 *The assumption $\hat{G}(0,0) < \infty$ implies that the right side of* (5.51) *is strictly larger than 1. Thus,* (5.49) *must hold when $\hat{G}(0,0) < \infty$ and $\mathbf{P}[W = 1] = 1$.*

Proof. Note that $\gamma_t = P_t$ since $\mathbf{E}[W] = 1$. By (5.40)

$$\mathbf{E}[\eta_t^2(k)] = (P_t\eta_0(k))^2 + \mathbf{E}[(N_t^t(k))^2]. \tag{5.53}$$

It is the second term on the right side above that requires the most effort. By (5.49) and (5.16)

$$\mathbf{E}[N_t^t(k)^2] =$$
$$\sum_{i,j}\int_0^t p_{t-s}(k,i)p_{t-s}(k,j)\sum_l \mathbf{E}[((Wp)-I)(l,i)((W\tilde{p})-I)(l,j)]\mathbf{E}[\eta_s^2(l)]\, ds.$$

By summing on k, we obtain

$$\mathbf{E}[\langle (N_t^t)^2, \mathbf{1}\rangle] = \sum_i \int_0^t \mathbf{E}[((Wp)-I)\hat{p}_{2(t-s)}((W\tilde{p})-I)](i,i)\mathbf{E}[\eta_s^2(i)]\, ds, \tag{5.54}$$

We define

$$\hat{p}(t) = \mathbf{E}[((Wp)-I)\hat{p}_t((W\tilde{p})-I)](i,i), \qquad i \in \mathbb{Z}^d,$$

which by translation invariance of p and $\hat{p}_t$ does not depend on i. Note that

$$\begin{aligned}\int_0^\infty \hat{p}(t)dt &= \mathbf{E}[((Wp)-I)\hat{G}((W\tilde{p})-I)](0,0)\\ &= \mathbf{E}[W^2](p\hat{G}\tilde{p})(0,0) - \hat{G}(0,0) + 2,\end{aligned} \tag{5.55}$$

and that, by assumption, $\int_0^\infty \hat{p}(t)dt < 2$. Hence

$$\begin{aligned}\int_0^T \mathbf{E}[\langle \eta_t^2, \mathbf{1}\rangle]\, dt &= \int_0^T \langle (p_t\eta_0)^2, \mathbf{1}\rangle\, dt + \int_0^T dt \int_0^t ds\, \hat{p}(2(t-s))\mathbf{E}[\langle \eta_s^2, \mathbf{1}\rangle]\\ &\le \frac{1}{2}\left(\langle \eta_0, \hat{G}\eta_0\rangle + \int_0^\infty \hat{p}(t)dt \int_0^T \mathbf{E}[\langle \eta_t^2, \mathbf{1}\rangle]\, dt\right),\end{aligned} \tag{5.56}$$

and we get

$$\int_0^T \mathbf{E}[\langle \eta_t^2, \mathbf{1}\rangle]\, dt \le \frac{\langle \eta_0, \hat{G}\eta_0\rangle}{2 - \int_0^\infty \hat{p}(t)\, dt} < \infty. \tag{5.57}$$

Now let $T \to \infty$. In particular

$$\int_0^\infty \mathbf{E}[\langle \eta_t^2, \mathbf{1}\rangle]\, dt \le \frac{\hat{G}(0,0)}{\hat{G}(0,0) - \mathbf{E}[W^2](p\hat{G}\tilde{p})(0,0)}\, \langle \eta_0, \mathbf{1}\rangle^2 < \infty. \tag{5.58}$$

□

To sum up, we now have the following.

Theorem 5.4 *Let η_t be the smoothing process. Assume either that $\mathbf{P}[W=1]<1$, or that $\mathbf{P}[W=1]=1$, (5.39) holds and $\hat{G}(0,0)<\infty$. If $\mathcal{L}[\eta_0]\in\mathcal{M}_\theta$, or $p(i,j)$ is a random walk kernel and $\mathcal{L}[\eta_0]$ is translation invariant, shift ergodic and satisfies $\mathbf{E}[\eta_0(0)]=\theta$, then $\mathcal{L}[\eta_t]\Rightarrow\nu_\theta$ as $t\to\infty$.*

Finally we note that our method fails for the smoothing process with $\mathbf{P}[W=1]=1$. First of all it is clear that we could not get (5.49) as we did above. In fact, we can give an example where (5.49) turns out to be wrong. Hence in this case we could not apply any version of Theorem 3.1. Here is the example.

Consider the *nearest neighbour* smoothing process η_t and potlatch process $\tilde{\eta}_t$ on $\mathbb{Z}$ with $\mathbf{P}[W=1]=1$. Hence, if $A(i,j)$ is given by (5.48), $\int(A-I)(i,k)(A-I)(j,l)d\mu=\delta(k,l)(p-I)(k,i)(p-I)(l,j)$ assumes the values

$$\begin{cases} 1, & i=j=k=l, \\ -\frac{1}{2}, & i\sim j=k=l, \\ -\frac{1}{2}, & j\sim i=k=l, \\ \frac{1}{4}, & j\sim k=l\sim i, \end{cases} \tag{5.59}$$

where $i\sim j$ means $|i-j|=1$. In particular, $q(\mathbf{1}_{\{k\}};i,i+2)=\frac{1}{4}\mathbf{1}_{k=i+1}$. It follows that

$$\int_0^\infty\langle\mathbf{1},|q(\tilde{\eta}_t)|\mathbf{1}\rangle\,dt\geq\frac{1}{4}\int_0^\infty\langle\tilde{\eta}_t^2,\mathbf{1}\rangle\,dt. \tag{5.60}$$

Note that for the potlatch process, $\langle\tilde{\eta}_t,\mathbf{1}\rangle=\langle\tilde{\eta}_0,\mathbf{1}\rangle=1$ $\mathbf{P}^{\delta_0}$-a.s., $t\geq 0$, and $\mathbf{E}^{\delta_0}[\tilde{\eta}_t(u)]=\gamma_t(0,u)$, where $\gamma_t=P_t$, the probability transition function of nearest-neighbour, rate-one, continuous random walk on $\mathbb{Z}$. Let $\rho_t=\mathbf{1}_{(-t^{3/4},t^{3/4})}$. By Chebyshev's inequality, there exists a $C\in(0,\infty)$ such that

$$\mathbf{E}^{\delta_0}[\langle\tilde{\eta}_t,\mathbf{1}-\rho_t\rangle]\leq C/t^{3/2}$$

On account of this estimate,

$$\int_0^\infty\mathbf{E}^{\delta_0}[\langle\tilde{\eta}_t,\mathbf{1}-\rho_t\rangle]dt<\infty.$$

By the Borel-Cantelli lemma, it follows that

$$\lim_{n\to\infty}\int_n^{n+1}\langle\tilde{\eta}_t,\mathbf{1}-\rho_t\rangle=0\qquad\mathbf{P}^{\delta_0}-a.s. \tag{5.61}$$

From this one can show, using the bounded transition rates of the potlatch process, that $\lim_{t\to\infty}\langle\tilde{\eta}_t,\mathbf{1}-\rho_t\rangle=0$ $\mathbf{P}^{\delta_0}$-a.s., and thus

$$\lim_{t\to\infty}\langle\tilde{\eta}_t,\rho_t\rangle=1\qquad\mathbf{P}^{\delta_0}\text{–a.s.} \tag{5.62}$$

Since $\langle\tilde{\eta}_t,\rho_t\rangle^2\leq\langle\tilde{\eta}_t^2,\mathbf{1}\rangle\langle\mathbf{1},\rho_t\rangle$, and $\langle\mathbf{1},\rho_t\rangle\sim 2t^{3/4}$ as $t\to\infty$,

$$\int_1^\infty\langle\tilde{\eta}_t^2,\mathbf{1}\rangle\,dt\geq\int_1^\infty\frac{\langle\tilde{\eta}_t,\rho_t\rangle^2}{\langle\mathbf{1},\rho_t\rangle}\,dt=\infty\quad\mathbf{P}^{\delta_0}\text{–a.s..}$$

by (5.62). Thus, (5.49) does not hold.

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