PATHWISE CONVERGENCE OF A RESCALED SUPER-BROWNIAN CATALYST REACTANT PROCESS

KLAUS FLEISCHMANN, ACHIM KLENKE, AND JIE XIONG

Abstract. Consider the one-dimensional catalytic super-Brownian motion X (called the reactant) in the catalytic medium ϱ which is an autonomous classical super-Brownian motion.

We characterize (ϱ, X) both in terms of a martingale problem and (in dimension one) as solution of a certain stochastic partial differential equation.

The focus of this paper is for dimension one the analysis of the longtime behavior via a mass-time-space rescaling. When scaling time by a factor of K, space is scaled by K^{η} and mass by $K^{-\eta}$. We show that for every parameter value $\eta \geq 0$ the rescaled processes converge as $K \to \infty$ in *path space*. While the catalyst's limiting process exhibits a phase transition at $\eta = 1$, the reactant's limit is always the same degenerate process.

Date: September 7, 2004, revised April 21, 2005.

1991 *Mathematics Subject Classification.* Primary 60K35; Secondary 60G57, 60J80.

Key words and phrases. Catalyst, reactant, superprocess, martingale problem, stochastic equation, density field, collision measure, collision local time, extinction, critical scaling, convergence in path space.

Partly supported by the program of the German Science Foundation: "Interacting Stochastic Systems of High Complexity", by an NSA research grant and by a Fellowship of the Alexander von Humboldt Foundation (Xiong).

1. Introduction and main results

1.1. **Background and sketch of results.** Super-Brownian motion ϱ is a measure-valued process in $\mathbb{R}^d$ where, loosely speaking, infinitesimal particles undergo Brownian motion and critical branching (at a constant rate $\gamma_c > 0$). Rigorous definitions have been given by three different means:

(i) as a multiplicative measure-valued Markov process whose log-Laplace transition functionals solve a certain integral equation (see, e.g., [Daw93, Chapter 4]),

(ii) as the solution of a martingale problem (MP) (see, e.g., [Daw93, Chapter 6] or [Eth00, Chapter 1.5]),

(iii) in the case $d = 1$, the states of ϱ have Lebesgue densities that can be characterized as the weak solution of a stochastic partial differential equation (SPDE) (see, e.g., [KS88] or [Rei89]).

In this paper we consider catalytic super-Brownian motion X, called the *reactant*, which is a process in the random medium ϱ, henceforth called the *catalyst*. Given a realization of ϱ, the infinitesimal X-particles undergo Brownian motion and critical binary branching with the only difference that the local branching rate is now nonconstant and is proportional to “the local density of ϱ”. The latter can be made precise in terms of the collision local time of a Brownian motion with ϱ (see [BEP91]). For $d \leq 3$, this collision local time is not identically 0 and is a continuous additive functional of Brownian motion. Using the collision local time, Dawson and Fleischmann constructed X in [DF97a], where it was characterized by its log-Laplace functionals.

For surveys on catalytic branching models we refer to [DF00, DF02] or [Kle00].

The first aim of this paper is to give a characterization of X in terms of a martingale problem and, for $d = 1$, by an SPDE. Uniqueness in the martingale problem will be obtained by an approximate duality in the spirit of [Myt98] and [Wan98]. For the SPDE we rely on recent results of [Zäh05].

The main goal of this work, however, is to investigate in $d = 1$ the longtime behavior of the bivariate process (ϱ, X) via a mass-time-space rescaling procedure. Assume that the initial states are chosen as $\varrho_0 = i_c\,\ell$ and $X_0 = i_r\,\ell$ (with ℓ the normalized Lebesgue measure and $i_c, i_r > 0$ constants). It is well known that (see [DF97a], see also the appendix of this paper) that

$$(\varrho_t, X_t) \underset{t\to\infty}{\longrightarrow} (0, i_r\,\ell) \quad \text{in probability} \tag{1.1}$$

(based on the topology of vague convergence of measures). Now we try to catch a richer structure on a larger scale. For this purpose, for a fixed scaling index $\eta \geq 0$, we rescale a measure-valued path $\mu = \{\mu_t : t \geq 0\}$ on $\mathbb{R}$ by

$$^{K}\mu_t(B) := K^{-\eta}\,\mu_{Kt}(K^\eta B), \qquad t \geq 0, \quad \text{Borel } B \subseteq \mathbb{R}, \quad K \geq 1. \tag{1.2}$$

By a result of Dawson and Fleischmann (see [DF88, Theorem 3.1]) the interesting scale for ϱ is $\eta = 1$. Here the limit of the $^{K}\varrho$ is non-trivial and can be considered as “super-Brownian without motion”. More precisely, the limit is the measure-valued process where the mass in each interval evolves as Feller's continuous state branching diffusion (the solution Z of the stochastic differential equation $dZ_t = \sqrt{\gamma_c Z_t}\,\mathrm{d}B_t$) and the evolutions in disjoint intervals are independent. For any positive time t the state of the limiting process is a random measure whose distribution is compound Poisson. More precisely, the states have a representation as $\int_{\mathbb{R}\times(0,\infty)} Y(\mathrm{d}(x,\alpha))\,\alpha\,\delta_x$, where Y is a Poisson point process on $\mathbb{R}\times(0,\infty)$ with intensity measure $i_{\mathrm{c}}(\gamma_{\mathrm{c}}t)^{-2}\ e^{-\alpha/(\gamma_{\mathrm{c}}t)}\,\mathrm{d}x\,\mathrm{d}\alpha$. In particular, for $t>0$ the states of this process are purely atomic and the positions of the atoms form a Poisson point process with intensity proportional to t^{-1}. Clearly this implies that for $\eta<1$ any possible limit of ${}^{K}\!\varrho$ is constantly 0 while a law of large numbers implies that for $\eta>1$ that the limit must be constantly $i_{\mathrm{c}}\,\ell$. For the reactant X, a similar averaging argument yields convergence of ${}^{K}\!X$ to a process constantly $i_{\mathrm{r}}\,\ell$ for any $\eta\geq 0$. The main point of this paper is thus to show that for both processes convergence happens not only in finite dimensional distributions but also in path space.

1.2. **Notation.** For $\lambda\in\mathbb{R}$ let $\phi_\lambda(x):=e^{-\lambda|x|}$, $x\in\mathbb{R}^d$, and define for $f:\mathbb{R}^d\to\mathbb{R}$ the norm $|f|_\lambda:=\|f/\phi_\lambda\|_\infty$, where $\|\cdot\|_\infty$ is the supremum norm. Denote by $\mathcal{C}_\lambda$ the (separable) Banach space of all continuous functions $f:\mathbb{R}^d\to\mathbb{R}$ with $|f|_\lambda<\infty$ and such that $f(x)/\phi_\lambda(x)$ has a finite limit as $|x|\to\infty$. Introduce the spaces

$$\mathcal{C}_{\exp}:=\bigcup_{\lambda>0}\mathcal{C}_\lambda \quad\text{and}\quad \mathcal{C}_{\mathrm{tem}}:=\bigcap_{\lambda>0}\mathcal{C}_{-\lambda} \tag{1.3}$$

of *exponentially decreasing* and *tempered* continuous functions on $\mathbb{R}^d$, respectively. We also need the space $\mathcal{C}_{\mathrm{com}}$ of all continuous functions on $\mathbb{R}^d$ with compact support.

Write $\mathcal{C}_\lambda^{(m)}$, $\mathcal{C}_{\exp}^{(m)}$, and $\mathcal{C}_{\mathrm{com}}^{(m)}$ if we additionally require that all partial derivatives up to the order $m\geq 1$ exist and belong to $\mathcal{C}_\lambda$, $\mathcal{C}_{\exp}$ and $\mathcal{C}_{\mathrm{com}}$, respectively.

Note that $\mathcal{C}_{\mathrm{tem}}$ equipped with the metric

$$d(f,g):=\sum_{n=1}^{\infty}2^{-n}\left(|f-g|_{-1/n}\ \wedge\ 1\right),\qquad f,g\in\mathcal{C}_{\mathrm{tem}}\,, \tag{1.4}$$

is a Polish space.

Let $\mathcal{M}$ denote the set of all (non-negative) Radon measures μ on $\mathbb{R}^d$ and let d_0 a complete metric on $\mathcal{M}$ which induces the vague topology. If μ has a Lebesgue density we denote also this density by μ. Write $\langle\mu,f\rangle$ for the integral of the function f with respect to the measure μ. Further equip the space

$$\mathcal{M}_{\mathrm{tem}}:=\left\{\mu\in\mathcal{M}:\ \langle\mu,\phi_\lambda\rangle<\infty \text{ for all } \lambda>0\right\}$$

with the metric

$$d_{\mathrm{tem}}(\mu,\nu):=d_0(\mu,\nu)+\sum_{n=1}^{\infty}2^{-n}\left(|\mu-\nu|_{1/n}\wedge 1\right),\qquad \mu,\nu\in\mathcal{M}_{\mathrm{tem}}\,. \tag{1.5}$$

Here $|\mu-\nu|_\lambda$ is an abbreviation for $\big|\langle\mu,\phi_\lambda\rangle-\langle\nu,\phi_\lambda\rangle\big|$. Note that $(\mathcal{M}_{\mathrm{tem}}\,,d_{\mathrm{tem}})$ is a Polish space, and that $\mu_n\to\mu$ in $\mathcal{M}_{\mathrm{tem}}$ if and only if

$$\langle\mu_n\,,\varphi\rangle\ \underset{n\to\infty}{\longrightarrow}\ \langle\mu,\varphi\rangle\quad \text{for all }\ \varphi\in\mathcal{C}_{\exp}\,. \tag{1.6}$$

Let $\mathcal{C}([0,\infty),\mathcal{M})$ be the space of continuous functions $[0,\infty)\to\mathcal{M}$, topologized in the canonical way, and define $\mathcal{C}((0,\infty),\mathcal{M})$, $\mathcal{C}([0,\infty),\mathcal{M}_{\mathrm{tem}})$ and so on similarly.

Random objects are always thought of as being defined over a large enough stochastic basis $(\Omega,\mathcal{F},\mathcal{F}_\cdot,\mathcal{P})$ satisfying the usual hypotheses. If $Y=\{Y_t:\,t\geq 0\}$ is a random process, then as a rule the law of Y is denoted by P^Y. We use $\mathcal{F}_t^Y$ to denote the completion of the σ–field $\bigcap_{\varepsilon>0}\sigma\{Y_s:\ s\leq t+\varepsilon\}$, $t\geq 0$. Sometimes we write $\mathcal{L}(Y)$ and $\mathcal{L}(Y\,|\,\cdot)$ for the law and conditional law of Y, respectively.

Define the heat kernel $p_t(x) := (2\pi t)^{-d/2} \exp\Big(-\frac{|x|^2}{2t}\Big)$ for $t > 0$ and $x \in \mathbb{R}^d$. Finally, for any $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor = \max\{n \in \mathbb{Z} :\ n \le x\}$ the integer part of x.

1.3. **The Model.** Super-Brownian motion (with branching rate $\gamma_{\rm c} > 0$ and diffusion constant $\sigma_{\rm c} > 0$) is an $\mathcal{M}_{\rm tem}$–valued Markov process ϱ that is multiplicative in the sense

$$P^{\varrho}_{\mu+\nu} e^{-\langle \varrho_t,\varphi\rangle} = (P^{\varrho}_{\mu} e^{-\langle \varrho_t,\varphi\rangle}) \times (P^{\varrho}_{\nu} e^{-\langle \varrho_t,\varphi\rangle}), \qquad \mu,\nu \in \mathcal{M}_{\rm tem},\ t \ge 0.$$

Here $\varphi \in \mathcal{C}_{\rm com}$, $\varphi \ge 0$, is a test function and P^{ϱ}_{μ} refers to ϱ started in $\varrho_0 = \mu \in \mathcal{M}_{\rm tem}$. Further it has continuous paths and the finite-dimensional distributions are characterized by the log-Laplace transforms

$$v^{\varrho}(0,t,\varphi;x) := -\log P^{\varrho}_{\delta_x} e^{-\langle \varrho_t,\varphi\rangle}, \quad t \ge 0,$$

which are the unique non-negative solutions of the integral equation

$$v^{\varrho}(s,t,\varphi;x) = (p_{\sigma_{\rm c}(t-s)} * \varphi)(x) - \frac{\gamma_{\rm c}}{2}\int_s^t {\rm d}r \int_{\mathbb{R}} {\rm d}y\, p_{\sigma_{\rm c}(t-r)}(y-x) v^{\varrho}(r,t,\varphi;y)^2,$$

for $0 \le s \le t$. We use the time-inhomogeneous notation here as it is more transparent when it comes to the reactant process.

An alternative description is via a martingale problem. The process ϱ is the unique process such that (for suitably smooth test functions φ) the process

$$M^{\rm c}_t(\varphi) := \langle \varrho_t,\varphi\rangle - \langle \varrho_0,\varphi\rangle - \int_0^t {\rm d}s \Big\langle \varrho_s\,,\, \frac{\sigma_{\rm c}^2}{2}\Delta\varphi\Big\rangle, \qquad t \ge 0,$$

is a square-integrable continuous $\mathcal{F}^{\varrho}_{\cdot}$–martingale with square function

$$\big\langle\!\big\langle M^{\rm c}(\varphi)\big\rangle\!\big\rangle_t = \gamma_{\rm c} \int_0^t {\rm d}s\ \big\langle \varrho_s, \varphi^2\big\rangle, \qquad t \ge 0.$$

See [Daw93], [LG99], [Eth00] or [Per02] for reference for super-Brownian motion.

In dimension $d = 1$ the states ϱ_t of super-Brownian motion have a Lebesgue density that will also be denoted by ϱ_t. The third characterization of ϱ (see [KS88, Rei89]) is as the unique (weak) solution of the SPDE

$$\frac{\partial}{\partial t}\varrho_t(x) = \frac{\sigma_{\rm c}^2}{2}\Delta\varrho_t(x) + \sqrt{\gamma_{\rm c}\,\varrho_t(x)}\,\dot W^{\rm c}_t(x),$$

$t > 0$, $x \in \mathbb{R}$, where $\dot W^{\rm c}$ is a (standard) time-space white noise.

We come now to the description of the reactant process X in terms of log-Laplace transforms. Fix a diffusion constant $\sigma_{\rm r} > 0$ and a proportionality constant $\gamma_{\rm r} > 0$ for the branching rate. Assume that $d \le 3$. Fix a realization ϱ of the catalyst process. Write $P^{X|\varrho}$ for the conditional probability of the X-process given ϱ. Since all spaces are Polish, we can consider $P^{X|\varrho}$ as the regular conditional probabilities $\mathcal{P}(\,\cdot\,|\varrho)$ of some probability measure $\mathcal{P}$. We write $P^{X|\varrho}_{s,\mu}$ if X is started at time s in the state μ and abbreviate $P^{X|\varrho}_{\mu} = P^{X|\varrho}_{0,\mu}$. Given the realization ϱ, X is a multiplicative $\mathcal{M}_{\rm tem}$–valued time-inhomogeneous Markov process, i.e. for any $t \ge s \ge 0$ and any $\mu,\nu \in \mathcal{M}_{\rm tem}$

$$P^{X|\varrho}_{s,\mu+\nu}(e^{-\langle X_t,\varphi\rangle} \,\big|\, \varrho) = \big(P^{X|\varrho}_{s,\mu}(e^{-\langle X_t,\varphi\rangle} \,\big|\, \varrho)\big) \times \big(P^{X|\varrho}_{s,\nu} e^{-\langle X_t,\varphi\rangle} \,\big|\, \varrho\big).$$

For initial states $X_s = \delta_x$, its log-Laplace transforms

$$v^{X|\varrho}(s,t,\varphi;x) := -\log P^{X|\varrho}_{s,\delta_x}\big(e^{-\langle \varrho_t,\varphi\rangle} \,\big|\, \varrho\big), \quad t \ge s \ge 0$$

are the unique non-negative solutions of the integral equation

$$\begin{aligned} v^{X|\varrho}(s,t,\varphi;x) &= (p_{\sigma_{\rm r}(t-s)} * \varphi)(x) \\ &\quad - \frac{\gamma_{\rm r}}{2}\int_s^t \mathrm{d}r \int_{\mathbb{R}} \varrho_r(\mathrm{d}y)\, p_{\sigma_{\rm r}(t-r)}(x-y) v^{X|\varrho}(r,t,\varphi;y)^2, \end{aligned} \tag{1.7}$$

for $0 \le s \le t$. For ϱ_0 with a continuous density which is in $\mathcal{C}_{\rm tem}$ and for $X_0 \in \mathcal{M}_{\rm tem}$ the processes ϱ and X are well-defined (see [FK99, Proposition 5] for even slightly weaker conditions).

Definition 1.1. *Under these conditions, the pair (ϱ, X) defined above is called the catalyst-reactant process.*

Our first task is now to give a description of the bivariate process (ϱ, X) as the solution of a martingale problem and as the solution of an SPDE. In order to describe the quadratic variation process we will need the so-called collision local time of the reactant X with the catalyst ϱ. To this end we recall the notion of collision local time in a broader context from [BEP91].

Definition 1.2 (Collision local time)**.** *Let $\mathbf{Y} = \big(Y^1, Y^2\big)$ be an $(\mathcal{M}_{\rm tem})^2$–valued continuous process. A continuous non-decreasing $\mathcal{M}_{\rm tem}$–valued stochastic process $t \mapsto L_{\mathbf{Y}}(t) = L_{\mathbf{Y}}(t,\,\cdot\,)$ is called* collision local time *of $\mathbf{Y}$, if for all $t > 0$ and $\varphi \in \mathcal{C}_{\rm exp}$*

$$\big\langle L^\varepsilon_{\mathbf{Y}}(t), \varphi \big\rangle \longrightarrow \langle L_{\mathbf{Y}}(t), \varphi\rangle \quad \textit{as} \quad \varepsilon \downarrow 0 \quad \textit{in } \mathcal{P}\textit{–probability.} \tag{1.8}$$

Here the approximating collision local times $L^\varepsilon_{\mathbf{Y}}$ are defined by

$$\langle L^\varepsilon_{\mathbf{Y}}(t), \varphi\rangle := \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} Y^1_s(\mathrm{d}x) \int_{\mathbb{R}^d} Y^2_s(\mathrm{d}y)\; p_\varepsilon(x-y)\, \varphi\big(\tfrac{x+y}{2}\big). \tag{1.9}$$

The collision local time $L_{\mathbf{Y}}$ will also be considered as a (locally finite) measure $L_{\mathbf{Y}}(\mathrm{d}(s,x))$ on $[0,\infty)\times\mathbb{R}^d$. ◇

Note that uniqueness of the collision local time is clear while showing the existence can be a problem. In fact, existence may even fail.

Definition 1.3 (Martingale problem)**.** *A random element (ϱ, X) in $\mathcal{C}\big([0,\infty), \mathcal{M}^2_{\rm tem}\big)$ is a solution of the martingale problem (MP) with diffusion constants $\sigma_{\rm c}, \sigma_{\rm r} > 0$ and branching rates $\gamma_{\rm c}, \gamma_{\rm r} > 0$, if for any $\varphi^{\rm c}, \varphi^{\rm r} \in \mathcal{C}^{(2)}_{\rm exp}$, the processes*

$$M^{\rm c}_t(\varphi^{\rm c}) := \langle \varrho_t\,, \varphi^{\rm c}\rangle - \langle \varrho_0\,, \varphi^{\rm c}\rangle - \int_0^t \mathrm{d}s \Big\langle \varrho_s\,,\, \frac{\sigma_{\rm c}^2}{2}\Delta\varphi^{\rm c}\Big\rangle, \qquad t \ge 0, \tag{1.10}$$

and

$$M^{\rm r}_t(\varphi^{\rm r}) := \langle X_t\,, \varphi^{\rm r}\rangle - \langle X_0\,, \varphi^{\rm r}\rangle - \int_0^t \mathrm{d}s \Big\langle X_s\,,\, \frac{\sigma_{\rm r}^2}{2}\Delta\varphi^{\rm r}\Big\rangle, \qquad t \ge 0, \tag{1.11}$$

are orthogonal square-integrable continuous martingales with square functions

$$\big\langle\!\big\langle M^{\rm c}(\varphi^{\rm c})\big\rangle\!\big\rangle_t = \gamma_{\rm c}\int_0^t \mathrm{d}s\; \big\langle \varrho_s\,, (\varphi^{\rm c})^2\big\rangle, \qquad t \ge 0, \tag{1.12}$$

and

$$\big\langle\!\big\langle M^{\mathrm{r}}(\varphi^{\mathrm{r}})\big\rangle\!\big\rangle_t = \gamma_{\mathrm{r}}\,\big\langle L_{(\varrho,X)}(t),(\varphi^{\mathrm{r}})^2\big\rangle, \qquad t\ge 0. \tag{1.13}$$

Here $L_{(\varrho,X)}$ is the collision local time of ϱ and X and we assume that the convergence (1.8) even holds $\mathcal{P}$–almost surely. $\diamond$

Consider now the situation $d=1$.

Definition 1.4 (SPDE)**.** *A pair (ϱ,X) of random time-dependent non-negative functions on $\mathbb{R}$ is said to be a weak solution of the stochastic partial differential equation (SPDE)*

$$\begin{aligned} \frac{\partial}{\partial t}\varrho_t(x) &= \frac{\sigma_{\mathrm{c}}^2}{2}\Delta\varrho_t(x) + \sqrt{\gamma_{\mathrm{c}}\,\varrho_t(x)}\,\dot W^{\mathrm{c}}_t(x),\\ \frac{\partial}{\partial t}X_t(x) &= \frac{\sigma_{\mathrm{r}}^2}{2}\Delta X_t(x) + \sqrt{\gamma_{\mathrm{r}}\,\varrho_t(x)\,X_t(x)}\,\dot W^{\mathrm{r}}_t(x), \end{aligned} \tag{1.14}$$

$t>0$, $x\in\mathbb{R}$, if on some probability space there exists a pair of independent time-space white noises $(\dot W^{\mathrm{c}},\dot W^{\mathrm{r}})$ such that for any $\varphi^{\mathrm{c}},\varphi^{\mathrm{r}}\in\mathcal{C}_{\mathrm{exp}}$ and $t\ge 0$,

$$\begin{aligned} \langle\varrho_t,\varphi^{\mathrm{c}}\rangle &= \langle\varrho_0,\varphi^{\mathrm{c}}\rangle + \int_0^t \mathrm{d}s\,\Big\langle\varrho_s,\frac{\sigma_{\mathrm{c}}^2}{2}\varphi^{\mathrm{c}}\Big\rangle + \int_{[0,t]\times\mathbb{R}} \mathrm{d}W^{\mathrm{c}}_s(x)\ \sqrt{\gamma_{\mathrm{c}}\,\varrho_s(x)},\\ \langle X_t,\varphi^{\mathrm{r}}\rangle &= \langle X_0,\varphi^{\mathrm{r}}\rangle + \int_0^t \mathrm{d}s\ \Big\langle X_s,\frac{\sigma_{\mathrm{r}}^2}{2}\varphi^{\mathrm{r}}\Big\rangle + \int_{[0,t]\times\mathbb{R}} \mathrm{d}W^{\mathrm{r}}_s(x)\ \sqrt{\gamma_{\mathrm{r}}\,\varrho_s(x)\,X_s(x)} \end{aligned}$$

almost surely. $\diamond$

Theorem 1.5 (Description by MP and SPDE)**.** *Assume that $X_0\in\mathcal{M}_{\mathrm{tem}}$ and that ϱ_0 has a density that is in $\mathcal{C}_{\mathrm{tem}}$. Let (ϱ,X) be the catalyst-reactant process defined in Definition 1.1.*

(i) *The pair (ϱ,X) is the unique solution of the martingale problem (MP).*
(ii) *If $d=1$ and if X_0 has a density that is in $\mathcal{C}_{\mathrm{tem}}$, then ϱ and X have jointly (time-space) continuous density fields that are the unique weak solutions of (SPDE).*

1.4. **Scaling limits.** For $\eta>1$ define ${}^\infty\varrho$ by ${}^\infty\varrho_t := i_{\mathrm{c}}\,\ell$, $t\ge 0$. For $0\le\eta<1$ let ${}^\infty\varrho_t := 0$, $t>0$, ${}^\infty\varrho_0 := i_{\mathrm{c}}\,\ell$. However, for $\eta=1$ let ${}^\infty\varrho$ denote the continuous $\mathcal{M}_{\mathrm{tem}}$-valued process with the following properties:

- For $A\subset\mathbb{R}$ bounded and Borel, ${}^\infty\varrho(A)$ is Feller's branching diffusion with rate γ_{c} and initial value ${}^\infty\varrho_0(A) = i_{\mathrm{c}}\,\ell(A)$.
- For disjoint measurable sets $A_1,A_2,\ldots\subset\mathbb{R}$, the processes ${}^\infty\varrho(A_1),{}^\infty\varrho(A_2),\ldots$ are independent.

Clearly, ${}^\infty\varrho_t$ is atomic for $t>0$, and for $\varepsilon>0$ the following processes coincide in law:

$$\{{}^\infty\varrho_t:\ t\ge\varepsilon\} \overset{\mathcal{D}}{=} \left\{\int_{\mathbb{R}}\pi_\varepsilon(\mathrm{d}x)\ \zeta^\varepsilon_t(x)\delta_x:\ t\ge\varepsilon\right\}, \tag{1.15}$$

where π_ε is a Poissonian point field on $\mathbb{R}$ with intensity $\varepsilon^{-1}i_{\mathrm{c}}$ and $\zeta^\varepsilon = \{\zeta^\varepsilon(x): x\in\mathbb{R}\}$ is a family of independent Feller's branching diffusions (with rate γ_{c}) starting at time $t=\varepsilon$ from independent identically exponentially distributed variables $\{\zeta^\varepsilon_\varepsilon(x): x\in\mathbb{R}\}$ with mean ε.

For all $\eta\ge 0$, define ${}^\infty X_t := i_{\mathrm{r}}\,\ell$, $t\ge 0$. We can now formulate the main result of this paper. Recall the notation from (1.2).

Theorem 1.6 (Scaling limit)**.** *Assume* $(\varrho_0, X_0) = (i_{\mathrm{c}}\,\ell, i_{\mathrm{r}}\,\ell)$. *For all* $\eta \geq 0$,

$$({}^K\!\varrho, {}^K\!X) \underset{K\to\infty}{\longrightarrow} ({}^\infty\!\varrho, {}^\infty\!X) \text{ in law on } \mathcal{C}\big((0,\infty), \mathcal{M}^2_{\mathrm{tem}}\big). \tag{1.16}$$

For $\eta \geq 1$, (1.16) *also holds on* $\mathcal{C}([0,\infty), \mathcal{M}^2_{\mathrm{tem}})$.

Remark 1.7. Clearly, for $\eta < 1$ the limiting catalyst is discontinuous at 0 and thus convergence holds only in $\mathcal{C}\big((0,\infty), \mathcal{M}_{\mathrm{tem}}\big)$. $\diamond$

1.5. **Outline.** The structure of the paper is as follows. In Section 2 we show Theorem 1.5.

In Section 3, Theorem 1.6 is proved. A key step for the case $\eta < 1$ is to show that the catalyst ϱ dies out in any bounded region when K is large enough. Then tightness of the reactant can be shown by a suitable decomposition. In the case $\eta \geq 1$ (Section 3.5), the catalyst will not die in a bounded region. The crux in proving the tightness for the reactant here is to divide a bounded region into many subregions and to show that for large K the catalyst dies out in "most" of them.

2. Martingale problem and SPDE

The purpose of this section is to prove Theorem 1.5. For notational simplicity we will assume throughout the rest of the paper that

$$\gamma_{\mathrm{c}} = \gamma_{\mathrm{r}} = \sigma_{\mathrm{c}} = \sigma_{\mathrm{r}} = 1.$$

2.1. **Existence of a catalyst reactant pair.** For the case $d = 1$ both (i) and (ii) have been shown by [Zäh05] (see Proposition 2.5 and Theorem 1.3 and 1.4). The fact that, for $d \leq 3$, (ϱ, X) is a solution of (MP) follows from standard calculations just as in the case $d = 1$. We omit the details.

It remains to show uniqueness of the solution of (MP). Here we rely on a method of approximate duality (see [Myt98] or [Wan98]). Consider a catalyst reactant pair (ϱ, X). Recall that $\mathcal{F}^\varrho$ (respectively $\mathcal{F}^X$) are the natural right-continuous complete filtration induced by ϱ (respectively X). Let $\mathcal{F}^\varrho_\infty = \sigma(\mathcal{F}^\varrho_t,\, t \geq 0)$ and

$$\mathcal{G}_t := \mathcal{F}^X_t \vee \mathcal{F}^\varrho_\infty.$$

Clearly, for any $\varphi \in \mathcal{C}^{(2)}_{\exp}$, the process $M^{\mathrm{r}}(\varphi)$ is a $\mathcal{P}$-martingale with respect to $\mathcal{G}$ and is thus for any ϱ a $P^{X|\varrho}$–martingale. By standard arguments, for given ϱ this family extends to a square-integrable *martingale measure* $M^{\mathrm{r}}(\mathrm{d}(s,x))$ and to the usual class of predictable integrands (see [Wal86]). In particular, for $\psi \in \mathcal{C}^{(1,2)}_{T,\exp}$, $T > 0$,

$$\begin{aligned} \langle M^{\mathrm{r}}, \psi\rangle_t &:= \int_{[0,t]\times\mathbb{R}^d} M^{\mathrm{r}}\,(\mathrm{d}(s,x))\,\psi_s(x) \\ &= \langle X_t\,, \psi_t\rangle - \langle X_0\,, \psi_0\rangle - \int_0^t \mathrm{d}s \Big\langle X_s\,, \frac{\partial}{\partial s}\psi_s + \frac{1}{2}\Delta\psi_s\Big\rangle \end{aligned} \tag{2.1}$$

is a continuous martingale with square function

$$\langle\!\langle M^{\mathrm{r}}, \psi\rangle\!\rangle_t \;=\; \int_{[0,t]\times\mathbb{R}^d} L_{(\varrho,X)}(\mathrm{d}(s,x))\,\psi^2_s(x) \;=:\; \big\langle L_{(\varrho,X)}, \psi^2\big\rangle_t\,, \tag{2.2}$$

$0 \leq t \leq T$. Here $\mathcal{C}^{(1,2)}_{T,\exp} := \bigcup_{\lambda>0} \mathcal{C}^{(1,2)}_{T,\lambda}$ with $\mathcal{C}^{(1,2)}_{T,\lambda} = \mathcal{C}^{(1,2)}_{T,\lambda}\big([0,T]\times\mathbb{R}^d\big)$ the set of all (real-valued) functions ψ defined on $[0,T]\times\mathbb{R}^d$ such that $t \mapsto \psi(t,\,\cdot)$, $t \mapsto \dfrac{\partial}{\partial t}\psi(t,\,\cdot)$, and $t \mapsto \Delta\psi(t,\,\cdot)$ are continuous $\mathcal{C}_\lambda$–valued functions.

For $s \in [0,T]$ let $v_s(x) := v^{X|\varrho}(s,T,\varphi;x)$. For $\varepsilon \in (0,1]$ introduce the smoothed catalyst $\varrho_s^\varepsilon := p_\varepsilon * \varrho_s$ and denote by $v_s^\varepsilon(x) := v^{X|\varrho^\varepsilon}(s,T,\varphi;x)$ the unique solution to (1.7) with $\varrho_s(\mathrm{d}y)$ replaced by $\varrho_s^\varepsilon(y)\,\mathrm{d}y$. Assuming additionally $\varphi \in \mathcal{C}_{\exp}^{(2)}$, then v^ε belongs to $\mathcal{C}_{T,\exp}^{(1,2)}$ and is the unique non-negative solution to the partial differential equation related to (1.7),

$$-\frac{\partial}{\partial s}\, v_s^\varepsilon(x) = \frac{1}{2}\Delta\, v_s^\varepsilon(x) - \frac{1}{2}\,\varrho_s^\varepsilon(x)\,(v_s^\varepsilon)^2(x), \qquad 0 \le s \le T, \quad x \in \mathbb{R}^d, \tag{2.3}$$

with terminal condition $v_T^\varepsilon = \varphi$. Trivially, we have the uniform domination

$$0 \;\le\; v_s^\varepsilon(x) \;\le\; p_{T-s} * \varphi\,(x), \qquad 0 \le s \le T, \quad x \in \mathbb{R}^d. \tag{2.4}$$

Note that $\varrho_t^\varepsilon \to \varrho_t$ vaguely as $\varepsilon \to 0$. Hence, at least formally, v^ε should converge to the solution of (2.3) with ϱ^ε formally replaced by ϱ, which, by (1.7) is v. (A rigorous proof can be found in [DF97a], see Theorem 4 (page 259) and Proposition 1(b) (page 225).) Hence, we get

$$v_s^\varepsilon(x) \underset{\varepsilon\downarrow 0}{\longrightarrow} v_s(x), \qquad 0 \le s \le T, \quad x \in \mathbb{R}^d. \tag{2.5}$$

Entering v^ε into (2.1) and (2.2) in place of ψ, and using (2.3) gives

$$\mathrm{d}\langle M^{\mathrm{r}}, v^\varepsilon\rangle_s = \mathrm{d}\langle X_s\,, v_s^\varepsilon\rangle - \frac{1}{2}\,\big\langle X_s\,,\, \varrho_s^\varepsilon\,(v_s^\varepsilon)^2\big\rangle\,\mathrm{d}s \tag{2.6}$$

with square function

$$\mathrm{d}\langle\!\langle M^{\mathrm{r}}, v^\varepsilon\rangle\!\rangle_s \;=\; \mathrm{d}\big\langle L_{(\varrho,X)}, (v^\varepsilon)^2\big\rangle_s\,. \tag{2.7}$$

By Itô's formula, this implies

$$\mathrm{d}e^{-\langle X_s\,, v_s^\varepsilon\rangle} \;=\; e^{-\langle X_s\,, v_s^\varepsilon\rangle}\Big[-\,\mathrm{d}\langle M^{\mathrm{r}}, v^\varepsilon\rangle_s - \frac{1}{2}\big\langle X_s\,,\, \varrho_s^\varepsilon\,(v_s^\varepsilon)^2\big\rangle\mathrm{d}s + \frac{1}{2}\,\mathrm{d}\big\langle L_{(\varrho,X)}, (v^\varepsilon)^2\big\rangle_s\Big].$$

Hence, for each $\varepsilon \in (0,1]$,

$$\begin{aligned} P^{X|\varrho}\, e^{-\langle X_T\,,\varphi\rangle} \;=\; & P^{X|\varrho}\, e^{-\langle X_0\,, v_0^\varepsilon\rangle} - \frac{1}{2}\, P^{X|\varrho} \int_0^T \mathrm{d}s\, e^{-\langle X_s\,, v_s^\varepsilon\rangle}\big\langle X_s\,,\, \varrho_s^\varepsilon\,(v_s^\varepsilon)^2\big\rangle \\ & +\; \frac{1}{2}\, P^{X|\varrho} \int_{[0,T]\times\mathbb{R}^d} L_{(\varrho,X)}(\mathrm{d}(s,x))\;(v_s^\varepsilon)^2(x)\, e^{-\langle X_s\,, v_s^\varepsilon\rangle}. \end{aligned} \tag{2.8}$$

Since we assumed almost sure convergence of the sequence approaching $L_{(\varrho,X)}$, for each $f \in \mathcal{C}_{\exp}$ we have

$$\int_0^T \mathrm{d}s\,\big\langle X_s\,,\, \varrho_s^\varepsilon\, f\big\rangle \underset{\varepsilon\downarrow 0}{\longrightarrow} \big\langle L_{(\varrho,X)}(T), f\big\rangle, \quad \mathcal{P}\text{–almost surely}, \tag{2.9}$$

hence $P^{X|\varrho}$–almost surely, for P^ϱ–almost all ϱ. Thus, by the pointwise convergence of approximate solutions as in (2.5) and domination (2.4), the second and third term at the right hand side of (2.8) cancel each other as $\varepsilon \downarrow 0$. Therefore

$$P^{X|\varrho}\, e^{-\langle X_T\,,\varphi\rangle} = \lim_{\varepsilon\downarrow 0} P^{X|\varrho}\, e^{-\langle X_0\,, v_0^\varepsilon\rangle}, \qquad 0 \le \varphi \in \mathcal{C}_{\exp}^{(2)}, \tag{2.10}$$

(which is in fact $P^{X|\varrho}\, e^{-\langle X_0\,, v_0\rangle}$).

Summarizing, the Laplace functional of X_T with respect to $P^{X|\varrho}$ applied to all non-negative $\varphi \in \mathcal{C}_{\exp}^{(2)}$ is uniquely determined, hence the law of X_T, consequently the law of X with respect to $P^{X|\varrho}$ is uniquely determined ([EK86, 4.4.2]). Thus, (ϱ, X) coincides in law with the catalyst reactant pair (ϱ, X) of the previous subsection. This finishes the proof of Theorem 1.5.

3. Scaling limits

Here we prove Theorem 1.6. After adapting the martingale problems to the scaled processes, with Lemma 3.1 we prove tightness of the ${}^K\varrho$ under $\eta \geq 1$. With Corollary 3.3 we get extinction of ${}^K\varrho$ under $\eta < 1$ in a functional limit setting. The convergence in law ${}^KX_t \to {}^\infty X_t$ for fixed t and all $\eta \geq 0$ will be shown in Section 3.3 by a modification of the proof in the $\eta = 0$ case from [DF97a]. Tightness questions of the ${}^KX$ for $\eta < 1$ are dealt with in Lemma 3.6, whereas the most difficult case $\eta \geq 1$ is established by Lemma 3.7. Finally in Section 3.6 we assemble the previous results to complete the proof of Theorem 1.6.

3.1. **Tightness of the ${}^K\varrho$ in the case $\eta \geq 1$.** We start by observing that a simple calculation yields (for all $\eta \geq 0$) that for $\varphi^{\mathrm{c}}, \varphi^{\mathrm{r}} \in \mathcal{C}^{(2)}_{\exp}$, the processes defined by

$$
\begin{aligned}
M_t^{\mathrm{c},K}(\varphi^{\mathrm{c}}) &= \left\langle {}^K\varrho_t\,, \varphi^{\mathrm{c}} \right\rangle - \left\langle {}^K\varrho_0\,, \varphi^{\mathrm{c}} \right\rangle - K^{1-2\eta} \int_0^t \mathrm{d}s \left\langle {}^K\varrho_s\,, \frac{1}{2}\Delta\varphi^{\mathrm{c}} \right\rangle \\
M_t^{\mathrm{r},K}(\varphi^{\mathrm{r}}) &= \left\langle {}^KX_t\,, \varphi^{\mathrm{r}} \right\rangle - \left\langle {}^KX_0\,, \varphi^{\mathrm{r}} \right\rangle - K^{1-2\eta} \int_0^t \mathrm{d}s \left\langle {}^KX_s\,, \frac{1}{2}\Delta\varphi^{\mathrm{r}} \right\rangle,
\end{aligned}
\tag{3.1}
$$

are square-integrable continuous martingales with square functions

$$
\begin{aligned}
\left\langle\!\left\langle M^{\mathrm{c},K}(\varphi^{\mathrm{c}}) \right\rangle\!\right\rangle_t &= K^{1-\eta} \int_0^t \mathrm{d}s \left\langle {}^K\varrho_s\,, (\varphi^{\mathrm{c}})^2 \right\rangle, \qquad t \geq 0, \\
\left\langle\!\left\langle M^{\mathrm{r},K}(\varphi^{\mathrm{r}}) \right\rangle\!\right\rangle_t &= K^{1-\eta} \left\langle L_{({}^K\varrho, {}^KX)}(t), (\varphi^{\mathrm{r}})^2 \right\rangle, \qquad t \geq 0.
\end{aligned}
\tag{3.2}
$$

Recall that ${}^K\varrho_0 \equiv i_{\mathrm{c}}\,\ell$. As usual, we say that a family of random processes is tight, if their laws form a tight family.

Lemma 3.1 (Tightness of the ${}^K\varrho$ under $\eta \geq 1$)**.** *Under $\eta \geq 1$, the processes $\left\{{}^K\varrho :\ K \geq 1\right\}$ are tight in $\mathcal{C}([0,\infty), \mathcal{M}_{\mathrm{tem}})$.*

Proof. There exists a smoothed version $\tilde{\phi}_\lambda$ of ϕ_λ such that to each $\lambda \in \mathbb{R}$ and $m \geq 0$ there is a positive constant $c = c(\lambda, m)$ with the property

$$
c^{-1}\phi_\lambda(x) \;\leq\; \left| \frac{\mathrm{d}^m}{\mathrm{d}x^m}\, \tilde{\phi}_\lambda\,(x) \right| \;\leq\; c\phi_\lambda(x), \qquad x \in \mathbb{R},
\tag{3.3}
$$

(see, for instance, [Mit85, (2.1)]).

For the moment, fix $T, \lambda > 0$ and $q > 2$. Consider

$$
f_K(t) := \mathcal{P} \sup_{0 \leq s \leq t} \left[1 + \left\langle {}^K\varrho_s\,, \tilde{\phi}_\lambda \right\rangle \right]^q, \qquad K \geq 1, \quad 0 \leq t \leq T.
$$

Using the martingale (3.1), the estimate (3.3) in the case $m = 2$, the fact that $(a+b)^q \leq 2^{q-1}(a^q + b^q)$ for $a, b \geq 0$, and assuming $\eta \geq 1$, we get (with a different c)

$$
f_K(t) \;\leq\; c \left[f_K(0) + \mathcal{P}\left(\int_0^t \mathrm{d}s \left\langle {}^K\varrho_s\,, \tilde{\phi}_\lambda \right\rangle \right)^q + \mathcal{P} \sup_{0 \leq s \leq t} \left| M_s^{\mathrm{c},K}(\tilde{\phi}_\lambda) \right|^q \right],
\tag{3.4}
$$

$0 \leq t \leq T$, $K \geq 1$. By Burkholder's inequality, from (3.2) and by $\eta \geq 1$ we get for some $c_q < \infty$

$$
\mathcal{P} \sup_{0 \leq s \leq t} \left| M_s^{\mathrm{c},K}(\tilde{\phi}_\lambda) \right|^q \;\leq\; c_q\, \mathcal{P}\left(\int_0^t \mathrm{d}s \left\langle {}^K\varrho_s\,, (\tilde{\phi}_\lambda)^2 \right\rangle \right)^{q/2}.
$$

Using $\tilde{\phi}_\lambda \leq c(\lambda)$ and the simple inequality $|a| \leq 2^{-1/2}(1+a^2)$, $a \in \mathbb{R}$, we may continue inequality (3.4) with

$$f_K(t) \;\leq\; c\, f_K(0) \,+\, c' \int_0^t \mathrm{d}s\, f_K(s) \tag{3.5}$$

$0 \leq t \leq T$, $K \geq 1$, with a constant $c' = c'(\lambda, q, T)$. Then Gronwall's inequality gives

$$\sup_{K\geq 1} \mathcal{P} \sup_{0\leq s\leq T} \big\langle {}^K\!\varrho_s\,,\, \tilde{\phi}_\lambda \big\rangle^q \;\leq\; c'' \tag{3.6}$$

with a constant $c'' = c''(\lambda, q, T)$. Again from the martingale (3.1), for $\varphi \in \mathcal{C}_\lambda^{(2)}$, $\lambda > 0$, and $0 \leq t' \leq t \leq T$,

$$\begin{aligned}\mathcal{P}\left|\big\langle {}^K\!\varrho_t - {}^K\!\varrho_{t'}\,,\,\varphi\big\rangle\right|^q \;\leq\; c'''\, \mathcal{P}\bigg(\int_{t'}^t \mathrm{d}s\; \big\langle {}^K\!\varrho_s\,,\,\tilde{\phi}_\lambda\big\rangle\bigg)^q \\ +\, c'''\, \mathcal{P}\bigg(\int_{t'}^t \mathrm{d}s\; \big\langle {}^K\!\varrho_s\,,(\tilde{\phi}_\lambda)^2\big\rangle\bigg)^{q/2}\end{aligned}$$

with $c''' = c'''(q, \lambda)$. By (3.6) we may continue with

$$\sup_{K\geq 1} \mathcal{P}\left|\big\langle {}^K\!\varrho_t - {}^K\!\varrho_{t'}\,,\,\varphi\big\rangle\right|^q \;\leq\; c''''\,|t-t'|^{q/2}, \qquad 0 \leq t, t' \leq T, \tag{3.7}$$

with $c'''' = c''''(\lambda, q, T)$.

It is easy to finish the tightness proof by taking a $q > 2$ and exploiting [EK86, Theorem 3.9.1]. In fact, we use the relatively compact subsets

$$K((c_n)_{n\geq 1}) := \Big\{\mu \in \mathcal{M}_{\mathrm{tem}} :\; \langle \mu, \tilde{\phi}_{1/n}\rangle \leq c_n\,,\;\; n \geq 1\Big\} \;\subseteq\; \mathcal{M}_{\mathrm{tem}}$$

with $(c_n)_{n\geq 1}$ a sequence of positive numbers. Given $0 < \varepsilon \leq 1$, by (3.6), we can find $(c_n)_{n\geq 1}$ such that

$$\mathcal{P}\big({}^K\!\varrho_t \in K((c_n)_{n\geq 1}) \;\text{ for all }\; t \in [0,T]\big) \;\geq\; 1 - \varepsilon.$$

Then by (3.7), for $\varphi \in \mathcal{C}_{\exp}^{(2)}$, the families of random processes $t \mapsto \langle {}^K\!\varrho_t\,, \varphi\rangle$ restricted to $[0,T]$ are tight in $\mathcal{C}([0,T],\mathbb{R})$. Then by [EK86, Theorem 3.9.1] the tightness claim follows. (Note that all of our processes are continuous, thus the tightness in Skorohod space implies the tightness in our $\mathcal{C}$–space.) This finishes the proof. $\square$

3.2. **Extinction of ${}^K\!\varrho$ under $\eta < 1$.** This extinction will follow from the following strong local extinction property of one-dimensional super-Brownian motion ϱ starting from a Lebesgue measure.

Proposition 3.2 (Almost sure local finite time extinction of ϱ)**.** *For all bounded Borel sets $B \subset \mathbb{R}$, and $0 \leq \eta < 1$, P_ℓ^ϱ–almost surely,*

$$\varrho_T(T^\eta B) \,=\, 0, \quad \textit{for all sufficiently large } T. \tag{3.8}$$

Consequently almost surely, at all late times there will be no catalytic mass in a given "parabola", which is also consistent with the picture from the $\eta = 1$ scaling limit in Theorem 1.6, saying that at late times T clusters are of order T apart.

Proof. We adapt a method occurring in [DF97a, Section 6.2] and in [GKW99, Lemma 4.2] for the corresponding particle system. We may assume that B is a centered "ball" of radius $r \geq 1$, say. Using the branching property, we decompose

$\varrho = \sum_{i\in\mathbb{Z}} \varrho^i$ in independent copies ϱ^i of ϱ, but where ϱ^i starts from $\varrho_0^i = \mathbb{1}_{[i,i+1)}\,\ell$, i.e. from the Lebesgue measure restricted to $[i,i+1)$, $i\in\mathbb{Z}$.

For the moment, fix i such that $|i|\geq 2r$. Consider the event

$$\varrho_t^i(t^\eta B)\;>\;0, \quad \text{for some} \quad t\geq 1, \tag{3.9}$$

which we denote by E^i. Under E^i there are two cases: Such a t satisfies $t\leq t_i := (|i|/2r)^{\,1/\eta}$, or $t>t_i$. If $t\leq t_i$, then ϱ^i gives mass to the centered ball with radius $|i|/2$ at some time after 1 (note that $|i|/2 = t_i^\eta r$). Call this event E_1^i. On the other hand, if $t>t_i$, then ϱ^i has to survive by time t_i. Call this event E_2^i. Consequently, $E^i\subseteq E_1^i\cup E_2^i$.

Now the event E_1^i has a probability bounded by $c\,|i|^{-2}$, see [Isc88, Theorem 1]. On the other hand, E_2^i has probability bounded by $c\,t_i^{-1} = c\,|i|^{-1/\eta}$, since the total mass process of ϱ^i is Feller's branching diffusion (see, for instance, [DFM00, formula (73)]).

Consequently, E^i has probability bounded by $c\left(|i|^{-2}+|i|^{-1/\eta}\right)$ which is summable in i with $|i|\geq 2r$. By Borel-Cantelli, E^i occurs only for finitely many $i\in\mathbb{Z}$. But these finitely many ϱ^i die in finite (random) time, that is, for them we have $\varrho_t^i = 0$ for all sufficiently large t, a.s. This gives

$$\varrho_t(t^\eta B) = \sum_{i\in\mathbb{Z}} \varrho_t^i(t^\eta B) = 0, \quad \text{for all sufficiently large } t, \quad \text{a.s.,}$$

that is the claim (3.8). □

Corollary 3.3 (Almost sure local finite scaling extinction of ${}^K\!\varrho$ under $\eta<1$)**.** *In the case $\eta<1$, for each $\varepsilon>0$, bounded Borel set $B\subset\mathbb{R}$, and $\delta\geq 0$ satisfying $\eta+\delta<1$,*

$$ {}^K\!\varrho_t(K^\delta B) = 0, \quad t\geq\varepsilon, \quad \textit{for all sufficiently large } K, \quad P_\ell^\varrho\textit{–a.s.} \tag{3.10}$$

In particular, as $K\to\infty$, the processes ${}^K\!\varrho$ converge in law to 0 in path space $\mathcal{C}((0,\infty),\mathcal{M})$.

Proof. Fix $\eta,\varepsilon,B,\delta$ as in the corollary. Set $\tilde\eta := \eta+\delta$. For $t\geq\varepsilon$ and $K\geq 1$,

$$ {}^K\!\varrho_t(K^\delta B) = K^{-\eta}\,\varrho_{Kt}\big((Kt)^{\tilde\eta}\,t^{-\tilde\eta}B\big).$$

But there is a bounded Borel set $\tilde B\subset\mathbb{R}$, such that $t^{-\tilde\eta}B\subseteq\tilde B$, for all $t\geq\varepsilon$. Therefore (3.10) follows from Proposition 3.2 with η,B replaced by $\tilde\eta,\tilde B$. Taking $\delta=0$, by the definition of the topology in $\mathcal{C}((0,\infty),\mathcal{M})$ this implies the convergence claim, since ε and B had been arbitrary. This finishes the proof. □

3.3. **Convergence of the finite dimensional distributions of ${}^K\!X$.** As we already know that $X_t \underset{t\to\infty}{\longrightarrow} i_{\mathrm{r}}\,\ell$ stochastically in the vague topology, it is easy to infer convergence of the finite dimensional distributions of the rescaled process towards the trivial limit.

Lemma 3.4 (Convergence of finite dimensional distributions)**.** *Fix $\eta\geq 0$. For $m\geq 1$, $0\leq t_1\leq\cdots\leq t_m$, and $\varphi_1,\dots,\varphi_m\in\mathcal{C}_{\exp}^+$,*

$$\mathcal{P}\exp\Big[-\sum_{i=1}^m \big\langle {}^K\!X_{t_i},\varphi_i\big\rangle\Big] \underset{K\to\infty}{\longrightarrow} \exp\Big[-i_{\mathrm{r}}\sum_{i=1}^m \langle\ell,\varphi_i\rangle\Big]. \tag{3.11}$$

Proof. The proof is particularly simple. It suffices to consider the case $m = 1$. We will consider only the case $\varphi = \varphi_1 := \lambda\,\mathbb{1}_{[0,1]}$ for $\lambda \geq 0$. The general case can be obtained via the usual approximation arguments.

For $\eta = 0$ or $t = 0$ there is nothing to show (see (1.1)). Assume now $\eta > 0$ and $t > 0$. Hence for any $\lambda > 0$, using Jensen's inequality and translation invariance

$$\begin{aligned}
\mathcal{P}e^{-\lambda\,\cdot\,{}^K\!X_t([0,1])} &= \mathcal{P}e^{-\lambda\,K^{-\eta}X_{Kt}([0,K^\eta])} \\
&\leq \frac{1}{\lfloor K^\eta\rfloor}\sum_{i=0}^{\lfloor K^\eta\rfloor-1}\mathcal{P}\,e^{-(\lfloor K^\eta\rfloor/K^\eta)\lambda\,X_{Kt}([i,i+1))} \\
&= \mathcal{P}\,e^{-(\lfloor K^\eta\rfloor/K^\eta)\lambda\,X_{Kt}([0,1))} \\
&\longrightarrow e^{-\lambda\,i_{\mathrm{r}}}, \quad \text{as } K\to\infty,
\end{aligned}$$

where we used in the last step that $X_t \to i_{\mathrm{r}}\,\ell$ (stochastically). On the other hand we have, again by Jensen's inequality, $\mathcal{P}e^{-\lambda\,\cdot\,{}^K\!X_t([0,1])} \geq e^{-\lambda\,\mathcal{P}\,{}^K\!X_t([0,1])} = e^{-\lambda\,i_{\mathrm{r}}}$. Thus $\lim_{K\to\infty}\mathcal{P}\,e^{-\lambda\,\cdot\,{}^K\!X_t([0,1])} = e^{-\lambda\,i_{\mathrm{r}}}$. □

3.4. **Tightness for ${}^K\!X$ under $\eta < 1$.** For translation invariant random measures with bounded intensities the concepts of convergence in $\mathcal{M}_{\mathrm{tem}}$ in law and in $\mathcal{M}$ in law coincide. In fact, assume the convergence in law $\mu_n \to \mu_0$ in $\mathcal{M}$ as $n\to\infty$ and that all these random measures are translation invariant and with intensities i_n, $n \geq 0$, bounded in n. Consider any $\varphi \in \mathcal{C}^+_{\mathrm{exp}}$. For $0 < \varepsilon \leq 1$, choose $\varphi_\varepsilon \in \mathcal{C}^+_{\mathrm{com}}$ such that $\langle \ell, |\varphi - \varphi_\varepsilon|\rangle < \varepsilon$. Then

$$\begin{aligned}
&\big|Pe^{-\langle\mu_n,\varphi\rangle} - Pe^{-\langle\mu_0,\varphi\rangle}\big| \\
&\quad\leq \big|Pe^{-\langle\mu_n,\varphi_\varepsilon\rangle} - Pe^{-\langle\mu_0,\varphi_\varepsilon\rangle}\big| + (i_n + i_0)\,\big\langle \ell, |\varphi-\varphi_\varepsilon|\big\rangle.
\end{aligned} \tag{3.12}$$

Letting first $n\to\infty$ and then $\varepsilon \downarrow 0$ gives the claim. Consequently, we can concentrate on a compactly supported test function where it is convenient. We may even assume that the test function is twice continuously differentiable.

In order to deal with tightness of the ${}^K\!X$ in the case $\eta < 1$, we will decompose them into two parts which will be handled separately. To this end we will work with the SPDE formulation of the rescaled processes

$$\begin{aligned}
{}^K\!\varrho_t(x) &= {}^K\!p_t * {}^K\!\varrho_0\,(x) \\
&\quad+ \int_{[0,t)\times\mathbb{R}} \mathrm{d}{}^K\!W^{\mathrm{c}}_s(y)\;{}^K\!p_{t-s}(y-x)\sqrt{K^{1-\eta}\,{}^K\!\varrho_s(y)}, \\
{}^K\!X_t(x) &= {}^K\!p_t * {}^K\!X_0\,(x) \\
&\quad+ \int_{[0,t)\times\mathbb{R}} \mathrm{d}{}^K\!W^{\mathrm{r}}_s(y)\,{}^K\!p_{t-s}(y-x)\sqrt{K^{1-\eta}\,{}^K\!\varrho_s(y)\;{}^K\!X_s(y)},
\end{aligned} \tag{3.13}$$

Here ${}^K\!p_t := p_{K^{1-2\eta}t}$ and ${}^K\!\dot W^{\mathrm{c}}_s(x) := K^{\frac{1+\eta}{2}}\;\dot W^{\mathrm{c}}_{Ks}(K^\eta x)$ and ${}^K\!\dot W^{\mathrm{r}}_s(x) := K^{\frac{1+\eta}{2}}\;\dot W^{\mathrm{r}}_{Ks}(K^\eta x)$ are standard white noises.

We make the following assumption:

Assumption 3.5 (Choice of parameters)**.** Fix $0 \leq \eta < 1$, $0 < 2\varepsilon < T$, as well as a non-vanishing $\varphi \geq 0$ in $\mathcal{C}^{(2)}_{\mathrm{com}}$. Choose $\delta \geq 0$ such that $1/2 - \eta < \delta < 1 - \eta$. Without loss of generality we assume that $\mathrm{supp}(\varphi) \subset B := (-1,1)$. ◇

For the moment, fix also $K \geq 1$. We want to decompose ${}^K\!X_t$ according to the effects of branching (i) from inside $K^\delta B$ or before time ε and (ii) from outside of $K^\delta B$ and after time ε. From (3.13), with $\mathrm{d}W := \mathrm{d}^K W^{\mathrm{r}}$ we get,

$$ {}^K\!X_t(x) = {}^K\!I_t(x) + {}^K\!J_t^{\varepsilon,t}(x), \qquad t > \varepsilon, \quad x \in \mathbb{R}, \tag{3.14} $$

where

$$ \begin{aligned} {}^K\!I_t(x) &:= {}^K\!p_{t-\varepsilon} * {}^K\!X_\varepsilon\,(x) \\ &\qquad + \int_{[\varepsilon,t)\times K^\delta B} \mathrm{d}W_s(y)\, {}^K\!p_{t-s}(y-x) \sqrt{K^{1-\eta}\, {}^K\!\varrho_s(y)\, {}^K\!X_s(y)}\,, \end{aligned} \tag{3.15} $$

and, for $\varepsilon < t \leq t'$,

$$ {}^K\!J_t^{\varepsilon,t'}(x) := \int_{[\varepsilon,t)\times K^\delta B^{\mathrm{c}}} \mathrm{d}W_s(y)\, {}^K\!p_{t'-s}(y-x) \sqrt{K^{1-\eta}\, {}^K\!\varrho_s(y)\, {}^K\!X_s(y)}\,. \tag{3.16} $$

Then we have the following decomposition:

$$ \big\langle {}^K\!X_t\,,\varphi\big\rangle = \big\langle {}^K\!I_t\,,\varphi\big\rangle + \big\langle {}^K\!J_t^{\varepsilon,t},\varphi\big\rangle, \qquad t \geq \varepsilon, \tag{3.17} $$

understanding the pairings in the obvious way. Our plan is to show tightness of the two terms separately, and, in fact, in the case of the first one, conditioned on ϱ. In the first case, we use the catalyst's a.s. extinction within a parabola, whereas in the second case second moment estimates suffice.

Lemma 3.6 (Tightness for ${}^K\!X$ under $\eta < 1$)**.** *Impose Assumption* 3.5.

(a) *Conditioned on* ϱ, *the processes* $\big\{t \mapsto \langle {}^K\!I_t,\varphi\rangle :\ K \geq 1\big\}$ *are tight in* $\mathcal{C}([2\varepsilon,T],[0,\infty))$.

(b) *The processes* $\big\{t \mapsto \langle {}^K\!J_t^{\varepsilon,t},\varphi\rangle :\ K \geq 1\big\}$ *are tight in* $\mathcal{C}([\varepsilon,T],[0,\infty))$.

Proof of Lemma 3.6(a). Since $\eta + \delta < 1$, by the Corollary 3.3, P^ϱ–almost surely,

$$ {}^K\!\varrho_s(y) \;=\; 0, \quad s \geq \varepsilon, \quad y \in K^\delta B, \quad K \geq K_0 = K_0(\varrho,\eta,\varepsilon,\varphi,\delta), \ \text{ say.} $$

Hence, for these K, the integral term in (3.15) vanishes. Thus,

$$ \big\langle {}^K\!I_t\,,\varphi\big\rangle = \big\langle {}^K\!X_\varepsilon\,,\, {}^K\!p_{t-\varepsilon} * \varphi\big\rangle \;=:\; {}^K Y_{t-\varepsilon}\,, \qquad \varepsilon < t \leq T. $$

Given ϱ, introduce the events

$$ {}^K\!E_N = {}^K\!E_N(\varrho,\varepsilon,T,\varphi) := \Big\{ \sup_{\varepsilon \leq t \leq 2T} {}^K Y_t \leq N \Big\}, \qquad N \geq 1. $$

By Markov's inequality, for the complement ${}^K\!E_N^{\mathrm{c}}$ of ${}^K\!E_N$,

$$ \mathcal{P}\big\{ {}^K\!E_N^{\mathrm{c}} \;\big|\; \varrho \big\} \;\leq\; N^{-1}\, \mathcal{P}\Big\{ \sup_{\varepsilon \leq t \leq 2T} \big\langle {}^K\!X_\varepsilon\,,\, {}^K\!p_t * \varphi\big\rangle \Big|\; \varrho \Big\}. \tag{3.18} $$

But with a constant $c = c(\varepsilon,T) < \infty$,

$$ {}^K\!p_t \;\leq\; (2T/\varepsilon)^{1/2}\, {}^K\!p_{2T} = c\, {}^K\!p_{2T}, \qquad \varepsilon \leq t \leq 2T. $$

Hence, inequality (3.18) can be continued with

$$ \leq\; N^{-1}\, c\, \mathcal{P}\Big\{ \big\langle {}^K\!X_\varepsilon\,,\, {}^K\!p_{2T} * \varphi\big\rangle \;\Big|\; \varrho \Big\} = N^{-1}\, c\, c_{\mathrm{r}}\, \|\varphi\|_1\,. $$

Thus, for each $\delta > 0$ we find an $N_0 = N_0(\delta,\varepsilon,T,\varphi)$ such that

$$ \sup_{K \geq K_0} \mathcal{P}\big\{ {}^K\!E_N^{\mathrm{c}} \;\big|\; \varrho \big\} \;\leq\; \delta, \qquad N \geq N_0\,. $$

On the other hand, on ${}^K\!E_N$, for $\varepsilon \le s \le t \le T$,

$$\big|{}^K Y_t - {}^K Y_s\big| \;\le\; \big\langle {}^K X_\varepsilon\,,\, |{}^K p_t - {}^K p_s| * \varphi\big\rangle\,. \tag{3.19}$$

By differentiation and comparison we get

$$\Big|\frac{\partial}{\partial r}\,p_r(x)\Big| \le \frac{2}{r}\,p_{2r}(x), \qquad r>0, \quad x\in\mathbb{R}. \tag{3.20}$$

Therefore,

$$\big|{}^K p_t(x) - {}^K p_s(x)\big| \;\le\; \int_s^t \mathrm{d}r\,\Big|\frac{\partial}{\partial r}\,{}^K p_r(x)\Big| \;\le\; c\int_s^t \mathrm{d}r\;p_{2r}(x),$$

$\varepsilon \le s \le t \le T$, $x\in\mathbb{R}$. Inserting this into (3.19), on ${}^K\!E_N$,

$$\big|{}^K Y_t - {}^K Y_s\big| \;\le\; 2\int_s^t \mathrm{d}r\,\big\langle {}^K X_\varepsilon\,, {}^K p_{2r} * \varphi\big\rangle = 2\int_s^t \mathrm{d}r\;{}^K Y_{2r} \;\le\; c\,N\,|t-s|,$$

$\varepsilon \le s \le t \le T$. Consequently, given ϱ and on ${}^K\!E_N$, the processes $\big\{{}^K Y_t:\ \varepsilon \le t \le T\big\}$, $K \ge K_0$, are equi-continuous ([Yos74, Section III.3]), hence the processes

$$\big\{\big\langle {}^K X_\varepsilon\,,\ {}^K p_{t-\varepsilon} * \varphi\big\rangle:\ 2\varepsilon \le t \le T+\varepsilon\big\}\,,\ \ K \ge K_0\,,$$

are also equi-continuous on ${}^K\!E_N$, given ϱ. But then the processes

$$\big\{\big\langle {}^K I_t\,,\varphi\big\rangle:\ 2\varepsilon \le t \le T\big\}\,,\ \ K \ge K_0\,,$$

are also equi-continuous on ${}^K\!E_N$, given ϱ. This then gives tightness of the family $\big\{\big\langle {}^K I_t\,,\varphi\big\rangle:\ 2\varepsilon \le t \le T\big\}\,,\ K \ge 1$, of processes, given ϱ, finishing the proof. □

Proof of Lemma 3.6(b). Here we proceed without conditioning on ϱ. It suffices to show that there is a constant $c = c(\varepsilon,\varphi,B,T,\eta,\delta)$ and a $K_0 = K_0(\delta)$ such that

$$\sup_{K\ge K_0}\,\mathcal{P}\,\big|\langle {}^K J_t^{\varepsilon,t},\varphi\rangle - \langle {}^K J_r^{\varepsilon,r},\varphi\rangle\big|^2 \;\le\; c\,|t-r|^2, \qquad r,t\in[\varepsilon,T]. \tag{3.21}$$

For this we may assume that $r<t$. We decompose

$$\langle {}^K J_t^{\varepsilon,t},\varphi\rangle - \langle {}^K J_r^{\varepsilon,r},\varphi\rangle = \langle {}^K J_t^{r,t},\varphi\rangle + \langle {}^K J^{\varepsilon,r,t},\varphi\rangle, \tag{3.22}$$

where

$$\begin{aligned}&{}^K J^{\varepsilon,r,t}(x)\\ &:= \int_{[\varepsilon,r)\times K^\delta B^{\mathrm{c}}} \mathrm{d}W_s(y)\,\big[{}^K p_{t-s}(y-x) - {}^K p_{r-s}(y-x)\big]\sqrt{K^{1-\eta}\,{}^K\varrho_s(y)\,{}^K X_s(y)}.\end{aligned}$$

Now we handle ${}^K J_t^{r,t}$ and ${}^K J^{\varepsilon,r,t}$ separately in order to prove (3.21).

1° (First term in the decomposition (3.22)). We show that there is a constant c and a $K_0 = K_0(\delta)$ such that (with $\zeta = \delta - \frac12 + \eta > 0$)

$$\sup_{K\ge K_0}\,K^{2\zeta}\mathcal{P}\big|\langle {}^K J_t^{r,t},\varphi\rangle\big|^2 \;\le\; c\,(t-r)^2. \tag{3.23}$$

First note that $t \mapsto \langle {}^K J_t^{r,t'},\varphi\rangle$, $\varepsilon \le t \le t' \le T$, is a continuous martingale and hence

$$\begin{aligned}&\mathcal{P}\big|\langle {}^K J_t^{r,t},\varphi\rangle\big|^2\\ &= \mathcal{P}\int_r^t \mathrm{d}s\int_{K^\delta B^{\mathrm{c}}}\mathrm{d}y\,\Big(\int_B \mathrm{d}x\,\varphi(x)\,{}^K p_{t-s}(y-x)\Big)^2 K^{1-\eta}\,{}^K\varrho_s(y)\,{}^K X_s(y).\end{aligned}$$

Note that $\mathcal{P}\,{}^K\!\varrho_s(y)\,{}^K\!X_s(y) \equiv i_{\rm c} i_{\rm r}$, and since ${}^K\!\varrho$ and ${}^K\!X$ are uncorrelated with expectations $\mathcal{P}^K\!\varrho_t(x) \equiv i_{\rm c}$ and $\mathcal{P}^K\!X_t(x) \equiv i_{\rm r}$. Let K_0 large enough such that $K_0^\delta \geq 2$. Hence for $K \geq K_0$ and $x \in B$ and $y \in K^\delta B^c$ we have $|y-x| \geq |y/2|$. Let $c' = 4\|\varphi\|_\infty^2 i_{\rm c} i_{\rm r}$ and assume $K \geq K_0$. Hence

$$\begin{aligned}
\mathcal{P}\big|\langle {}^K\!J_t^{r,t}, \varphi\rangle\big|^2 &\leq c' K^{1-\eta} \int_r^t \mathrm{d}s \int_{K^\delta B^c} \mathrm{d}y\, p_{K^{1-2\eta}(t-s)}(y/2)^2 \\
&= 2c' K^{1-\eta}(t-r)^{1/2} \int_0^1 \mathrm{d}s \int_{\frac12 K^\zeta (t-r)^{-1/2}}^\infty \mathrm{d}y\, p_s(y)^2 \\
&\leq 2c' K^{1-\eta}(t-r)^{1/2} \int_{\frac12 K^\zeta (t-r)^{-1/2}}^\infty \mathrm{d}y\, p_1(y)^2 \\
&\leq 2c' K^{1-\eta-2\zeta} T (t-r)^{1/2} \exp\left(-\frac14 K^{2\zeta}(t-r)^{-1}\right),
\end{aligned}$$

where we used a standard heat kernel estimate in the last step. Note that

$$\begin{aligned}
c'' &:= \sup_{0\leq r<t\leq T} (t-r)^{-3/2} \exp\left(-\frac18 (t-r)^{-1}\right) < \infty \\
c''' &:= \sup_{K\geq 1} K^{1-\eta-2\zeta} \exp\left(-\frac18 K^{2\zeta} T^{-1}\right) < \infty.
\end{aligned}$$

Hence, (3.23) holds with $c := 2c'\, c''\, c'''\, T$.

2° (Second term in the decomposition (3.22)). We show that there is a constant c and a K_0 such that

$$\sup_{K\geq K_0} \mathcal{P}\big|\langle {}^K\!J^{\varepsilon,r,t}, \varphi\rangle\big|^2 \;\leq\; c\,|t-r|^2 . \tag{3.24}$$

In fact, as in the first step we get that

$$\mathcal{P}\big|\langle {}^K\!J^{\varepsilon,r,t}, \varphi\rangle\big|^2 \;\leq\; c'\, K^{1-\eta} \int_\varepsilon^r \mathrm{d}s \int_{K^\delta B^c} \mathrm{d}y \Big(\int_B \mathrm{d}x\, \big|{}^K\!p_{t-s}(y-x) - {}^K\!p_{r-s}(y-x)\big| \Big)^2 .$$

Using the estimate $\big|\frac{\partial}{\partial\theta}{}^K\!p_\theta(y)\big| \leq 2p_{2\theta}(y)$ the dx–integral is bounded by

$$\int_B \mathrm{d}x \int_{r-s}^{t-s} \mathrm{d}\theta\, \Big|\frac{\partial}{\partial\theta}\, {}^K\!p_\theta(y-x)\Big| \leq\; 4 \int_{r-s}^{t-s} \mathrm{d}\theta\, \frac1\theta\, {}^K\!p_{2\theta}(y/2).$$

Hence the dy–integral is bounded by

$$c \int_{|y|\,\geq\, c\,K^\delta} \mathrm{d}y\, |t-r| \int_{r-s}^{t-s} \mathrm{d}\theta\, \frac{1}{\theta^2}\, {}^K\!p_{2\theta}^2(y/2).$$

Interchanging the order of integration and substituting $y \mapsto (8\,\theta\, K^{1-2\eta})^{1/2} y$ results into

$$c\,|t-r| \int_{r-s}^{t-s} \mathrm{d}\theta\, \frac{1}{\theta^2}\, \theta^{1/2}\, K^{-1/2+\eta} \int_{|y|\,\geq\, c\,K^{\eta+\delta-1/2}\,\theta^{-1/2}} \mathrm{d}y\, e^{-y^2}.$$

Using similar estimates as in the first step yields (3.24).

3° (Conclusion). Combining (3.23) and (3.24), by decomposition (3.22), claim (3.21) follows. This finishes the proof. □

3.5. **Pathwise convergence of the ${}^K\!X$ for $\eta \geq 1$.** Now we come to the key of the reactant's tightness proof.

Lemma 3.7 (Reactant's pathwise convergence for $\eta \geq 1$)**.** *Under $\eta \geq 1$,*

$$ {}^K\!X \to {}^\infty\!X \ \ \textit{in law on } \mathcal{C}(\mathbb{R}_+, \mathcal{M}) \ \textit{as} \ K \to \infty. \tag{3.25}$$

Proof. The concept of the proof is as follows: In Step 1 we take care of a small piece $[0, t_0]$ of the (macroscopic) time axis by a standard argument. In the subsequent steps we concentrate on $[t_0, \infty)$. In Steps 2 and 3 we construct a coupling with a reactant process that suffers killing at finitely many points $a \in A \subset \mathbb{R}$. In this new process the intervals between the points in A are decoupled reactant processes with killing at the boundaries. In Steps 4 and 5 we break the compact support of a test function into small intervals. We show that only a few of them are occupied by the catalyst ${}^K\!\varrho$ after time t_0. The boundary points of these intervals are essentially used to form the set A. In Step 6 we show that the original reactant process ${}^K\!X$ and the reactant with killing in A are close. We use the decoupling in Steps 7 and 8 to handle the pieces close to catalyst peaks and away from the catalyst's support separately and by different means.

Step 1 (Initial time interval). Fix a twice continuously differentiable compactly supported function $\varphi \geq 0$ on $\mathbb{R}$ and $\varepsilon > 0$.

Let $c_1 = \frac{2}{\varepsilon} i_{\rm r} 1 \langle \ell, |\Delta\varphi| \rangle$ and $c_2 = \frac{4}{\varepsilon^2} i_{\rm c} i_{\rm r} \langle \ell, \varphi^2 \rangle$. Hence for $t_0 > 0$ by Markov's inequality and Doob's inequality

$$\begin{aligned}
&\mathcal{P}\Big\{ \sup_{t\in[0,t_0]} \big| \langle {}^K\!X_t - i_{\rm r}\, \ell, \varphi \rangle \big| > \varepsilon \Big\} \\
&\quad \leq \mathcal{P}\Big\{ \frac{1}{2} K^{1-2\eta} \sup_{t\in[0,t_0]} \int_0^t \langle {}^K\!X_t, |\Delta\varphi| \rangle \, \mathrm{d}s \ > \frac{\varepsilon}{2} \Big\} + \mathcal{P}\Big\{ \sup_{t\in[0,t_0]} |M_t^{\mathrm{r},K}(\varphi)| > \frac{\varepsilon}{2} \Big\} \\
&\quad \leq c_1 t_0 K^{1-2\eta} + \frac{4}{\varepsilon^2} \mathcal{P}(M_{t_0}^{\mathrm{r},K}(\varphi))^2 \\
&\quad \leq c_1 t_0 K^{1-2\eta} + c_2 t_0 K^{1-\eta}.
\end{aligned}$$

For $\eta > 1$ this tends to 0 as $K \to \infty$ and hence the statement of the theorem is shown for this case. (However, as it requires only a minor modification of the argument, in the remainder we will consider both $\eta = 1$ and $\eta > 1$.) For all $\eta \geq 1$ we can choose $t_0 = \varepsilon/(2(c_1 + c_2))$, hence for $K \geq 1$

$$\mathcal{P}\Big\{ \sup_{t\in[0,t_0]} \big| \langle {}^K\!X_t - i_{\rm r}\, \ell, \varphi \rangle \big| > \varepsilon \Big\} < \ \varepsilon. \tag{3.26}$$

From now on we fix this t_0.

Step 2 (Killing at one point). Fix for the moment $K \geq 1$ and $a \in \mathbb{R}$. Define the reactant process ${}^K\!X^a$ as ${}^K\!X$ but with instantaneous killing of mass in the point a. We collect the killed "particles" in a second process ${}^K\!\tilde{X}^a$ (with zero initial state) and let them perform just the standard reactant process (with independent noise driving the SPDE). Thus ${}^K\!X \overset{\mathcal{D}}{=} \{ {}^K\!X_t^a + {}^K\!\tilde{X}_t^a : t \geq 0 \}$. It is easy to establish a coupling of these processes such that

$$ {}^K\!X_t \ = \ {}^K\!X_t^a + {}^K\!\tilde{X}_t^a, \qquad t \geq 0. \tag{3.27}$$

A construction can be done along the following lines: Let $\tau^a = \tau^a[{}^K\!\xi]$ denote the first hitting time of a Brownian motion ${}^K\!\xi$ with diffusion constant $K^{\frac{1}{2}-\eta}$, and let Π_x denote its laws when started in $x \in \mathbb{R}^d$. Then $V_t := ({}^K\xi_t,\, \mathbb{1}_{\tau^a \le t})$ is a Markov process on $\mathbb{R} \times \{0,1\}$ with càdlàg paths. Now construct the reactant process Z with motion process V and with catalyst (ϱ, ϱ). Then one can choose $Z\big(\,\cdot\, \times\{0\}\big)$ for X^a_t and $Z\big(\,\cdot\, \times\{1\}\big)$ for $\widetilde{X}^a_t$. We omit the details. In words, for the process Z thus constructed we have assigned each particle a type either 0 or 1. Initially all particles are of type 0. However, on hitting the site a, a particle changes its type to 1 which is kept forever. Particles of type 0 are considered as being alive while type 1 particles are zombie particles.

The important point to notice is that the total mass process

$$t \;\mapsto\; {}^K\tilde{M}^a_t \;:=\; \langle {}^K\tilde{X}^a_t, 1\rangle \tag{3.28}$$

is a non-negative continuous submartingale, and that by the reflection principle [since ${}^KX_0\big(\{a\}\big) = 0$],

$$P^{X|\varrho}\; {}^K\tilde{M}^a_t \;=\; \int_{\mathbb{R}} {}^KX_0(\mathrm{d}x)\; \Pi_x(\tau^a \le t) \;=\; 2\int_{\mathbb{R}} {}^KX_0(\mathrm{d}x) \int_{|x-a|}^{\infty} \mathrm{d}y\; {}^Kp_t(y). \tag{3.29}$$

Recall that ${}^Kp_t(y) = (2\pi K^{1-2\eta}t)^{-1/2}\exp(-y^2/2K^{1-2\eta}t)$. Since ${}^KX_0 = i_{\mathrm{r}}\,\ell$, we have

$$\begin{aligned} P^{X|\varrho}\; {}^K\tilde{M}^a_t \;&=\; 4i_{\mathrm{r}} \int_0^\infty \mathrm{d}x \int_x^\infty \mathrm{d}y\; (2\pi K^{1-2\eta}t)^{-1/2}\exp(-y^2/2K^{1-2\eta}t)\\ &=\; \frac{4i_{\mathrm{r}}}{\sqrt{2\pi K^{1-2\eta}t}} \int_0^\infty \mathrm{d}y\; y\exp(-y^2/2K^{1-2\eta}t)\\ &=\; \frac{4i_{\mathrm{r}}}{\sqrt{2\pi}}\sqrt{K^{1-2\eta}t}\,. \end{aligned} \tag{3.30}$$

If we fix a time horizon $T > 0$, then Doob's maximal inequality (e.g. [RY91, Theorem 2.1.7]) yields, for every $\delta > 0$,

$$\begin{aligned} P^{X|\varrho}\Big\{\sup_{t\in[0,T]} {}^K\tilde{M}^a_t > \delta\Big\} \;&\le\; \delta^{-1} P^{X|\varrho}\, {}^K\tilde{M}^a_T\\ &\le\; \frac{4i_{\mathrm{r}}}{\sqrt{2\pi}}\frac{\sqrt{T}}{\delta}\, K^{\frac12-\eta} \;\to\; 0 \;\text{ as }\; K\to\infty. \end{aligned} \tag{3.31}$$

Thus it will be sufficient to show convergence of ${}^KX^a$ instead of ${}^KX$.

Step 3 (Killing at finitely many points). Now consider a finite set $A \subset \mathbb{R}$. Hence as in Step 2 we can define a coupling

$${}^KX \;=\; {}^KX^A + {}^K\tilde{X}^A \tag{3.32}$$

with killing in all points $a \in A$. If we define

$${}^K\tilde{M}^A_t \;:=\; \big\langle {}^K\tilde{X}^A_t, 1\big\rangle, \qquad t \ge 0, \tag{3.33}$$

then again ${}^K\tilde{M}^A$, is a non-negative continuous submartingale. Clearly we have

$$P^{X|\varrho}\; {}^K\tilde{M}^A_T \;\le\; \sum_{a\in A} P^{X|\varrho}\; {}^K\tilde{M}^a_T\,. \tag{3.34}$$

Thus, by (3.31),

$$P^{X|\varrho}\Big\{\sup_{t\in[0,T]} {}^K\tilde{M}^A_t > \delta\Big\} \;\le\; |A|\,\frac{4i_{\mathrm{r}}}{\sqrt{2\pi}}\frac{\sqrt{T}}{\delta}\, K^{\frac12-\eta} \;\to\; 0 \;\text{ as }\; K\to\infty. \tag{3.35}$$

Hence it is sufficient to show convergence of ${}^K\!X^A$ instead of ${}^K\!X$, even if A depends on K and the number $|A|$ of points in A grows with K like $K^{\eta-1}$.

Step 4 (Equidistant decomposition). For our fixed φ, fix $L \geq 1$ such that $\varphi(x) = 0$ if $|x| > L$. Moreover, fix $T > t_0$. The strategy of the further proof is to split the interval $[-L, L]$, which supports the test function φ, into sufficiently small intervals $I_i = {}^K\!I_i$, $i = 0, \ldots, N-1$, $N = {}^K\!N \geq 1$, in such a way that most of the intervals are not populated by the catalyst ${}^K\!\varrho$ in the time interval $[t_0, T]$. Roughly speaking, the set A at which killing happens will be defined as the set of boundary points of the remaining intervals. This will allow a decoupling of the reactant ${}^K\!X^A$ in intervals where it simply follows the heat flow, and in intervals where also branching occurs. However the construction will be carried out in such a way that the total reactant mass involved in branching in $[t_0, T]$ is small and can easily be controlled.

In this step we start with breaking $[-L, L]$ into intervals and controlling the migration of catalyst into neighboring intervals.

Fix a number

$$\zeta \in \Big(\eta - 1, \eta - \frac{1}{2}\Big). \tag{3.36}$$

In the sequel we suppress the K–dependence in some quantities. Put

$$N \;=\; {}^K\!N \;:=\; \lfloor 2LK^\zeta \rfloor \tag{3.37}$$

(note that $N \to \infty$ as $K \to \infty$ by our assumption $\eta \geq 1$), and set

$$I_i \;=\; {}^K\!I_i \;:=\; \Big[-L + \frac{2L}{N} i,\; -L + \frac{2L}{N}(i+1)\Big], \qquad 0 \leq i < N. \tag{3.38}$$

By the branching property, we can assume a coupling

$${}^K\!\varrho \;=\; {}^K\!\check\varrho + \sum_{i=0}^{N-1} {}^K\!\varrho^i, \tag{3.39}$$

where $\{{}^K\!\check\varrho, {}^K\!\varrho^0, \ldots, {}^K\!\varrho^{N-1}\}$ is an independent family of (measure-valued) catalyst processes with initial conditions

$$\begin{aligned} {}^K\!\check\varrho_0 \;&=\; \mathbb{1}_{\mathbb{R}\setminus[-L,L]}\, i_{\mathrm{c}}\, \ell, \\ {}^K\!\varrho^i_0 \;&=\; \mathbb{1}_{{}^K\!I_i}\, i_{\mathrm{c}}\, \ell, \qquad 0 \leq i < N. \end{aligned} \tag{3.40}$$

(Note that the uniform initial states give no mass to common boundary points.)

Define $\check F = {}^K\!\check F$ and $F^i = {}^K\!F^i$, $0 \leq i < N$, as the events

$$\begin{aligned} \check F \;&:=\; \big\{{}^K\!\check\varrho_t(I_i) = 0,\;\; t \in [0,T],\;\; 1 \leq i < N-1\big\}, \\ F^i \;&:=\; \big\{{}^K\!\varrho^i_t(I_j) = 0\,,\;\; t \in [0,T],\;\; 0 \leq i, j < N,\;\; |j-i| \geq 2\big\}, \end{aligned}$$

and put

$$F \;=\; {}^K\!F \;:=\; \check F \cap \bigcap_{i=0}^{N-1} F^i. \tag{3.41}$$

That is, on the event F the "particles" of the catalyst ${}^K\!\varrho$ *cross at most one interval* by time T (including the two outside intervals). It is well-known, see, e.g., [Daw93, Theorem 9.2.4], replacing there R, by $K^{\eta-\zeta}$ and t, by KT, that there exists a constant $c = c(L, T, i_{\rm c})$ such that

$$\mathcal{P}(F^i) \;\geq\; 1 \,-\, c\,K^{\eta-2\zeta-3/2}\exp(-c\,K^{2(\eta-\zeta)-1}).$$

Thus, since $2(\eta-\zeta)-1 > 0$ by our assumption (3.36) on ζ,

$$\mathcal{P}\Big(\bigcap_{i=0}^{N-1} F^i\Big) \;\geq\; 1 - 2LK^{\zeta}\,c\,K^{\eta-2\zeta-3/2}\exp(-c\,K^{2(\eta-\zeta)-1}) \;\to\; 1 \text{ as } K\to\infty.$$

Similarly, decomposing the infinite initial measure and replacing R in the application of the former theorem by $nK^{\eta-\zeta}$, $n \geq 1$, one gets the well-known result that $\mathcal{P}({}^K\tilde{F}) \to 1$ as $K\to\infty$. Concluding, we have for $T > t_0 > 0$ (and the fixed L),

$$\mathcal{P}({}^KF) \to 1 \text{ as } K\to\infty. \tag{3.42}$$

Step 5 (Sparse occupation by the catalyst). In this step, we show that at time t_0 not too many of the intervals ${}^KI_i$, from (3.38) are occupied by the catalyst ${}^K\varrho$. Together with the result of the previous step this bounds the number of intervals touched by ${}^K\varrho$, between time t_0 and T.

For our fixed t_0 and L, define a set E, of indices by

$$E \;=\; {}^KE \;:=\; \Big\{i\in\{1,\dots,N-2\}:\; {}^K\varrho^i_{t_0}\neq 0\Big\}\cup\big\{0, N-1\big\},$$

with $N = {}^KN$ from (3.37). Note that by the survival probability of Feller's branching diffusion and the small starting mass there exists a constant $c = c(L, t_0, i_{\rm c}) > 0$ such that

$$\mathcal{P}(i\in E) \;\leq\; cK^{\eta-\zeta-1}, \qquad 1\leq i < N-1. \tag{3.43}$$

Opposed to (3.38), for $0\leq i<N$, set

$$\hat{I}_i \;=\; {}^K\hat{I}_i \;:=\; \Big[-L+\frac{2L}{N}(i-1),\; -L+\frac{2L}{N}(i+2)\Big] \tag{3.44}$$

and

$$C \;=\; {}^KC \;:=\; \bigcup_{i\in E}\hat{I}_i, \qquad D \;=\; {}^KD \;:=\; [-L,L]\setminus C. \tag{3.45}$$

Hence, on the event F, from (3.41) we have ${}^K\varrho_t(D) = 0$, $t\in[t_0,T]$.

By estimate (3.43) and the definition (3.37) of ${}^KN$, we get

$$\mathcal{P}|E| \;\leq\; 2 + c\,{}^KNK^{\eta-\zeta-1} \;\leq\; cK^{\eta-1}, \tag{3.46}$$

since $\eta\geq 1$. Thus

$$\mathcal{P}\ell({}^KC) \;\leq\; 3\,\frac{2L}{N}\,c\,K^{\eta-1} \;\leq\; c\,K^{\eta-\zeta-1} \;\to\; 0 \text{ as } K\to\infty, \tag{3.47}$$

by the definition (3.37) of ${}^KN$, and our assumption (3.36) on ζ. Hence, together with estimate (3.46) and (3.42), for our fixed $\varepsilon>0$, there exists a constant $K_0 = K_0(\varepsilon, t_0)$ and an $n = n(\varepsilon) > 0$ such that for all $K\geq K_0$,

$$\mathcal{P}(B) \;\geq\; 1-\varepsilon,$$

where

$$B \;=\; {}^KB \;=\; {}^KB(\varepsilon,n) \;:=\; \big\{|E|\leq nK^{\eta-1}\big\}\cap\Big\{\ell({}^KC)\leq\frac{\varepsilon^2}{\|\varphi\|_\infty\, i_{\rm r}}\Big\}\cap F. \tag{3.48}$$

Step 6 (Reactant's approximation by the process with killing). Now we define the set A, at which the reactant's killing takes place, as the boundary of the intervals that could be occupied by the catalyst between time t_0 and T :

$$A \;=\; {}^K\!A \;:=\; \bigcup_{i\in E,\ 0\le j\le 3} \Big\{ -L + \frac{2L}{N}(i-1+j) \Big\} \;\supseteq\; \partial C. \tag{3.49}$$

Clearly, by the definition (3.48) of $B = {}^K\!B$, we have $B \subseteq \big\{ |A| \le 4nK^{\eta-1} \big\}$. Hence, recalling the coupling (3.32) and (3.33), the estimate in (3.35) yields that on ${}^K\!B$, for $\delta > 0$,

$$P^{X|\varrho} \Big\{ \sup_{t\in[0,T]} \big\langle {}^K\!X_t - {}^K\!X^A_t, 1 \big\rangle > \delta \Big\} \;\le\; \frac{16\, n\, i_{\rm r} \sqrt{T}}{\delta \sqrt{2\pi}}\, K^{-1/2}.$$

Thus, for $K_0 = K_0(\varepsilon, n, t_0)$ sufficiently large, we have for all $K \ge K_0$ on ${}^K\!B$

$$P^{X|\varrho} \Big\{ \sup_{t\in[0,T]} \big\langle {}^K\!X_t - {}^K\!X^A_t, 1 \big\rangle > \frac{\varepsilon}{\|\varphi\|_\infty} \Big\} \;<\; \varepsilon \tag{3.50}$$

(with our fixed φ). This shows that it suffices henceforth to consider ${}^K\!X^A$ instead of ${}^K\!X$.

Step 7 (Killed process close to the catalyst's peaks). Recall the sets ${}^K\!C$, and ${}^K\!D$, where branching might be possible, and where the catalyst is absent, respectively. By construction, given ϱ, the processes $\{{}^K\!X^A_t\, \mathbb{1}_C \,:\, t \ge 0\}$ and $\{{}^K\!X^A_t\, \mathbb{1}_D \,:\, t \ge 0\}$ decouple. That is, they are independent reactant processes with killing in each point $a \in A = {}^K\!A$, [recall (3.49)]. Hence they can be treated separately. In this step we handle ${}^K\!X^A\, \mathbb{1}_C$,, and in the next step we will treat ${}^K\!X^A\, \mathbb{1}_D$.

The key is, that the initial mass of ${}^K\!X^A\, \mathbb{1}_C$ is small (as C is small). Clearly, by the killing, the total mass process $\big\{ {}^K\!X^A_t(C) \,:\, t \ge 0 \big\}$ is a non-negative right-continuous supermartingale. Hence by a variant of Doob's maximal inequality (e.g. [RY91, Exercise 2.1.15]) we get that on ${}^K\!B$

$$\begin{aligned} P^{X|\varrho} \Big\{ \sup_{t\in[t_0,T]} {}^K\!X^A_t(C) > \frac{\varepsilon}{\|\varphi\|_\infty} \Big\} \;&\le\; \frac{\|\varphi\|_\infty}{\varepsilon}\, P^{X|\varrho}\, {}^K\!X^A_{t_0}(C) \\ &\le\; \frac{\|\varphi\|_\infty}{\varepsilon}\, i_{\rm r}\, \ell(C) \;<\; \varepsilon \end{aligned}$$

(recall (3.48)). Thus, on ${}^K\!B$,

$$P^{X|\varrho} \Big\{ \sup_{t\in[t_0,T]} \big\langle {}^K\!X^A_t,\, \varphi\, \mathbb{1}_C \big\rangle > \varepsilon \Big\} \;<\; \varepsilon. \tag{3.51}$$

Step 8 (Killed process away from the catalyst's peaks). Since for ϱ, given, ${}^K\!X_{t_0}$ converges in probability to $i_{\rm r}\, \ell$ as $K \to \infty$ (Lemma 3.4), from (3.50) and (3.51) we get that on a subset $B' \subseteq {}^K\!B$ with $\mathcal{P}(B') > 1 - 2\varepsilon$, and for $K \ge K_0$ (where K_0 is sufficiently large),

$$P^{X|\varrho} \Big\{ \Big| \big\langle {}^K\!X^A_{t_0}\, \mathbb{1}_D - i_{\rm r}\, \ell,\, \varphi \big\rangle \Big| > 2\varepsilon \Big\} \;<\; 2\varepsilon. \tag{3.52}$$

Now $\{{}^K\!X^A_t\, \mathbb{1}_D : t \in [t_0, T]\}$ is heat flow (with speed $K^{\frac12 - \eta}$) with absorption in each point $a \in {}^K\!A$. Of course, (3.50) holds also if the supremum there is taken only on

$[t_0, T]$, if the initial state at time t_0 is ${}^K\!X^A_{t_0} \mathbb{1}_D$, and if the constant function $\mathbb{1}$ is replaced by $\mathbb{1}_D$. That is, for $K \geq K_0$,

$$P^{X|\varrho}\Big\{ \sup_{t\in[t_0,T]} \Big\langle ({}^K\!X^A_{t_0} \mathbb{1}_D) * {}^K\!p_{t-t_0} - {}^K\!X^A_t \mathbb{1}_D\,,\ 1 \Big\rangle > \frac{\varepsilon}{\|\varphi\|_\infty} \Big\} \;<\; \varepsilon. \tag{3.53}$$

On the other hand, as φ is continuous, we can choose $K_0 = K_0(\varepsilon)$ so large that

$$\sup_{K\geq K_0} \sup_{t\in[t_0,T]} \big\| \varphi - \varphi * {}^K\!p_{t-t_0} \big\|_\infty \;<\; \frac{\varepsilon^2}{2L\, i_{\mathrm{r}}}\,.$$

Hence, for $t \in [t_0, T]$ and $K \geq K_0$,

$$\begin{aligned} P^{X|\varrho}&\left\{ \sup_{t\in[t_0,T]} \left| \Big\langle ({}^K\!X^A_{t_0} \mathbb{1}_D) * {}^K\!p_{t-t_0} - {}^K\!X^A_{t_0} \mathbb{1}_D\,,\ \varphi \Big\rangle \right| > \varepsilon \right\} \\ &\leq\; \frac{\varepsilon}{2L\, i_{\mathrm{r}}} \mathcal{P}\, {}^K\!X_{t_0}(D) \;\leq\; \varepsilon \end{aligned} \tag{3.54}$$

since $D \subseteq [-L, L]$. Combining (3.50), (3.51), (3.53), (3.54), and (3.52), we have for $K \geq K_0$ on B' [with $\mathcal{P}(B') > 1 - 2\varepsilon$],

$$P^{X|\varrho}\Big\{ \sup_{t\in[t_0,T]} \Big| \big\langle {}^K\!X_t - i_{\mathrm{r}}\, \ell, \varphi \big\rangle \Big| > 3\varepsilon \Big\} \;<\; 3\varepsilon.$$

Hence for $K \geq K_0$,

$$\mathcal{P}\Big\{ \sup_{t\in[t_0,T]} \Big| \big\langle {}^K\!X_t - i_{\mathrm{r}}\, \ell, \varphi \big\rangle \Big| > 4\varepsilon \Big\} \;<\; 6\varepsilon.$$

This concludes the proof. □

3.6. **Completion of the proof of Theorem 1.6 .** Now we are ready to complete the proof of Theorem 1.6.

Case 1 (Catalyst, $\eta \geq 1$). For $\eta = 1$, the convergence to and the identification of the limit ${}^\infty\!\varrho$ of the ${}^K\!\varrho$ was provided in [DF88] (with a slightly different reference function and using a Skorohod space, but note that all of our processes are continuous). Hence for $\eta > 1$ the large of large numbers yields that ${}^K\!\varrho_t \longrightarrow i_{\mathrm{c}}\, \ell$ (stochastically) as $K \to \infty$ for all $t > 0$. By Lemma 3.1 the processes ${}^K\!\varrho$, $K \geq 1$, are tight in $\mathcal{C}([0,\infty), \mathcal{M}_{\mathrm{tem}})$. Hence convergence to the constant process is proved also in path space.

Case 2 (Catalyst, $\eta < 1$). The extinction of ${}^K\!\varrho$ under $\eta < 1$ on the path space $\mathcal{C}((0,\infty), \mathcal{M})$ was verified in Corollary 3.3.

Case 3 (Fdd convergence of ${}^K\!X$). For all $\eta \geq 0$, the convergence of one-dimensional distributions of the ${}^K\!X$ was provided by Lemma 3.4.

Case 4 (Reactant, $\eta < 1$). In the case $\eta < 1$, it is enough to show convergence in law ${}^K\!X \to {}^\infty\!X$ as $K \to \infty$ on function space $\mathcal{C}([2\varepsilon, T], \mathcal{M})$, for any choice of $0 < 2\varepsilon < T$. For this purpose, for fixed $\varphi \in \mathcal{C}^{(2)}_{\mathrm{com}}$, we can decompose as in (3.17):

$$\big\langle {}^K\!X_t\,, \varphi \big\rangle = \big\langle {}^K\!I_t\,, \varphi \big\rangle + \big\langle {}^K\!J^{\varepsilon,t}_t, \varphi \big\rangle, \qquad t \;\geq\; 2\varepsilon, \tag{3.55}$$

By Lemma 3.6(b), the second part forms a tight family of processes in $\mathcal{C}([2\varepsilon, T], \mathbb{R})$. Moreover, by (3.23), for fixed t,

$$\mathcal{P}\left| \langle {}^K\!J^{\varepsilon,t}_t, \varphi \rangle \right|^2 \;\leq\; c\, K^{-2\zeta} \underset{K\to\infty}{\longrightarrow} 0.$$

Therefore,

$$\left\langle {}^K J_{\cdot}^{\varepsilon,t}, \varphi \right\rangle \underset{K\to\infty}{\longrightarrow} 0 \quad \text{on function space.} \tag{3.56}$$

On the other hand, for fixed t, the term at the left hand side of (3.55) convergence in law to the required deterministic limit $\left\langle {}^\infty X_t\,, \varphi \right\rangle$. Therefore also the first term at the right hand side of (3.55) converges fdd to that limit. Hence, the P^ϱ–random finite dimensional distributions of the processes $t \mapsto \left\langle {}^K I_t\,, \varphi \right\rangle$ conditioned on ϱ converge in law to the ones of $\delta_{\langle {}^\infty X_\cdot\,, \varphi\rangle}$. Then by the conditioned tightness in Lemma 3.6(a), the P^ϱ–random distributions of the processes $\left\langle {}^K I_\cdot\,, \varphi \right\rangle$ converge in law to $\delta_{\langle {}^\infty X_\cdot\,, \varphi\rangle}$. Integrating out ϱ, the processes $\left\langle {}^K I_\cdot\,, \varphi \right\rangle$ converge in law to $\left\langle {}^\infty X_\cdot\,, \varphi \right\rangle$.

Putting this together with (3.56), by the decomposition (3.22) the processes $t \mapsto \left\langle {}^K X_t\,, \varphi \right\rangle$ converge in law to $\left\langle {}^\infty X_\cdot\,, \varphi \right\rangle$ on function space $\mathcal{C}([2\varepsilon, T], \mathbb{R})$. Since φ was arbitrary, the proof of Case 4 is finished.

Case 5 (Reactant, $\eta \geq 1$). This was provided in Section 3.5.

This completes the proof of Theorem 1.6 □

Acknowledgment. Part of this work was accomplished during the first author's visits to the UTK and the third author's visits to the WIAS. The hospitality and financial support is gratefully acknowledged. We also thank the referee for his comments which helped improving the exposition.

Appendix

In [DF97a] it was shown in Theorem 6 that $(\varrho_t, X_t) \underset{t\to\infty}{\longrightarrow} (0, i_{\mathrm{r}}\, \ell)$ in probability, which we need in the present paper in (1.1). As some people feel that the proof of Theorem 6 in [DF97a] is a bit short (and since the statement is a corner stone in the study here), we provide a few details for that proof here. In order to keep things short, we stick to the notation in [DF97a] and use it without further notice.

The last display formula in the proof of Theorem 6 (on page 274) is

(A.1)

$$\|v(s,t)\|_1 \geq \int \ell(\mathrm{d}a)\, \Pi_{s,a}\, \varphi(W_t)\, \exp\left[-(t-\tau_{s,a})_+^{-1/2} \|\varphi\|_1 \int_s^{\tau_{s,a}} \mathrm{d}r\, \rho_r(W_r) \right].$$

Here $v(s,t,a)$ is the log-Laplace transform of the reactant started at time s in δ_a, evaluated with respect to the nonnegative (compactly supported continuous) test function φ conditional on the catalyst ρ. Furthermore W is a Brownian motion started at time s in $a \in \mathbb{R}$ and $\tau_{s,a}$ is the last time where W collides with ρ. This time is shown to be finite. Since $\|v(s,t)\|_1 \leq \|\varphi\|_1$ by Jensen's inequality, it suffices to show that $\mathbb{P}_\ell \|v(s,t)\|_1 \to \|\varphi\|_1$ as $t \to \infty$. From (A.1) we get

$$\mathbb{P}_\ell \|v(s,t)\|_1 \geq \int \ell(\mathrm{d}a)\, \Pi_{s,a} \varphi(W_t)\, \mathbb{P}_l Y_t(W),$$

where $Y_t(W) = \exp\big[-(t-\tau_{s,a})_+^{-1/2} \|\varphi\|_1 \int_s^{\tau_{s,a}} \mathrm{d}r\, \rho_r(W_r) \big]$. By spatial translation invariance of ρ under $\mathbb{P}_\ell$ we get that $\mathbb{P}_l Y_t(W) = \mathbb{P}_\ell Y_t((W_r - a)_{r\geq s})$. Hence

$$\mathbb{P}_\ell \|v(s,t)\|_1 \geq \int \ell(\mathrm{d}a)\, \Pi_{s,0}\, \varphi(W_t + a)\, \mathbb{P}_l Y_t(W) = \|\varphi\|_1\, \Pi_{s,0}\, \mathbb{P}_l Y_t(W).$$

Now $Y_t(W) \to 1$, $\Pi_{s,0} \times \mathbb{P}_l$–almost surely as $t \to \infty$ and is bounded by 1. Hence $\Pi_{s,0} \mathbb{P}_l Y_t(W) \to 1$ as $t \to \infty$ and we are done.

References

[BEP91] M.T. Barlow, S.N. Evans, and E.A. Perkins. Collision local times and measure-valued processes. *Canad. J. Math.*, 43(5):897–938, 1991.

[Daw93] D.A. Dawson. Measure-valued Markov processes. In P.L. Hennequin, editor, *École d'été de probabilités de Saint Flour XXI–1991*, volume 1541 of *Lecture Notes in Mathematics*, pages 1–260. Springer-Verlag, Berlin, 1993.

[DF88] D.A. Dawson and K. Fleischmann. Strong clumping of critical space-time branching models in subcritical dimensions. *Stoch. Proc. Appl.*, 30:193–208, 1988.

[DF97a] D.A. Dawson and K. Fleischmann. A continuous super-Brownian motion in a super-Brownian medium. *Journ. Theoret. Probab.*, 10(1):213–276, 1997.

[DF97b] D.A. Dawson and K. Fleischmann. Longtime behavior of a branching process controlled by branching catalysts. *Stoch. Process. Appl.*, 71(2):241–257, 1997.

[DF00] D.A. Dawson and K. Fleischmann. Catalytic and mutually catalytic branching. In *Infinite Dimensional Stochastic Analysis*, pages 145–170, Amsterdam, 2000. Royal Netherlands Academy of Arts and Sciences.

[DF01] J.-F. Delmas and K. Fleischmann. On the hot spots of a catalytic super-Brownian motion. *Probab. Theory Relat. Fields*, 121(3):389–421, 2001.

[DF02] D.A. Dawson and K. Fleischmann. Catalytic and mutually catalytic super-Brownian motions. In *Ascona 1999 Conference*, volume 52 of *Progress in Probability*, pages 89–110. Birkhäuser Verlag, 2002.

[DFM00] D.A. Dawson, K. Fleischmann, and C. Mueller. Finite time extinction of superprocesses with catalysts. *Ann. Probab.*, 28(2):603–642, 2000.

[DFM02] D.A. Dawson, K. Fleischmann, and P. Mörters. Strong clumping of super-Brownian motion in a stable catalytic medium. *Ann. Probab.* 30(4):1990–2045, 2002.

[DFR91] D.A. Dawson, K. Fleischmann, and S. Roelly. Absolute continuity for the measure states in a branching model with catalysts. In *Stochastic Processes, Proc. Semin. Vancouver/CA 1990*, volume 24 of *Prog. Probab.*, pages 117–160, 1991.

[DP99] D.A. Dawson and E. Perkins. Measure-valued processes and renormalization of branching particle systems. In R.A. Camona and B. Rozovskii, editors, *Stochastic partial differential equations: six perspectives*, pages 45–106. Amer. Math. Soc., Providence, RI, 1999.

[Eth00] A. Etheridge. An introduction to superprocesses, *University lecture series, Vol. 20, Amer. Math. Soc.*, 2000.

[EF98] A.M. Etheridge and K. Fleischmann. Persistence of a two-dimensional super-Brownian motion in a catalytic medium. *Probab. Theory Relat. Fields*, 110(1):1–12, 1998.

[EK86] S.N. Ethier and T.G. Kurtz. *Markov Processes: Characterization and Convergence.* Wiley, New York, 1986.

[EP94] S.N. Evans and E.A. Perkins. Measure-valued branching diffusions with singular interactions. *Canad. J. Math.*, 46(1):120–168, 1994.

[FK99] K. Fleischmann and A. Klenke. Smooth density field of catalytic super-Brownian motion. *Ann. Appl. Probab.*, 9(2):298–318, 1999.

[FK00] K. Fleischmann and A. Klenke. The biodiversity of catalytic super-Brownian motion. *Ann. Appl. Probab.*, 10(4):1121–1136, 2000.

[FX01] K. Fleischmann and J. Xiong. A cyclically catalytic super-Brownian motion. *Ann. Probab.*, 29(2):820–861, 2001.

[GKW99] A. Greven, A. Klenke, and A. Wakolbinger. The longtime behavior of branching random walk in a random medium. *Electron. J. Probab.*, 4:no. 12, 80 pages (electronic), 1999.

[Isc88] I. Iscoe. On the supports of measure-valued critical branching Brownian motion. *Ann. Probab.*, 16:200–221, 1988.

[Kle00] A. Klenke. A review on spatial catalytic branching. In Luis G. Gorostiza and B. Gail Ivanoff, editors, *Stochastic Models*, volume 26 of *CMS Conference Proceedings*, pages 245–263. Amer. Math. Soc., Providence, 2000.

[Kot92] P. Kotelenez. Comparison methods for a class of function valued stochastic partial differential equations. *Probab. Theory Relat. Fields*, 93(1):1–29, 1992.

[KS88] N. Konno and T. Shiga. Stochastic partial differential equations for some measure-valued diffusions. *Probab. Theory Relat. Fields*, 79:201–225, 1988.

[LG99] J.-F. Le Gall. Spatial branching processes, random snakes and partial differential equations, *Lectures in Mathematics, ETH Zürich, Birkhäuser*, 1999.

[Mit85] I. Mitoma. An ∞–dimensional inhomogeneous Langevin equation. *J. Functional Analysis*, 61:342–359, 1985.

[Myt98] L. Mytnik. Weak uniqueness for the heat equation with noise. *Ann. Probab.*, 26(3):968–984, 1998.

[Per02] E.A. Perkins. Dawson-Watanabe Superprocesses and Measure-valued Diffusions. In P. Bernard, editor, *École d'été de probabilités de Saint Flour XXIX–1999*, volume 1781 of *Lecture Notes in Mathematics*, pages 125–334. Springer-Verlag, Berlin, 2002.

[Rei89] M. Reimers. One dimensional stochastic partial differential equations and the branching measure diffusion. *Probab. Theory Relat. Fields*, 81:319–340, 1989.

[RY91] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[Shi94] T. Shiga. Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Can. J. Math.*, 46(2):415–437, 1994.

[Wal86] J.B. Walsh. An introduction to stochastic partial differential equations. volume 1180 of *Lecture Notes Math.*, pages 266–439. École d'Été de Probabilités de Saint-Flour XIV – 1984, Springer-Verlag Berlin, 1986.

[Wan98] H. Wang. A class of measure-valued branching diffusions in a random medium. *Stochastic Anal. Appl.*, 6:753–786, 1998.

[Yos74] K. Yosida. *Functional Analysis.* Springer-Verlag, 4th edition, 1974.

[Zäh05] H. Zähle. On space-time reguarity of catalytic super-Brownian motion. *Mathem. Nachrichten (to appear)*, 2005.

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D–10117 Berlin, Germany

E-mail address: fleischm@wias-berlin.de
URL: http://www.wias-berlin.de/~fleischm

Johannes Gutenberg-Universität Mainz, Institut für Mathematik, Staudingerweg 9, 55099 Mainz, Germany

E-mail address: math@aklenke.de
URL: http://www.aklenke.de

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1300, USA. and Department of Mathematics, Hebei Normal University, Shijiazhuang 050016, P.R. China

E-mail address: jxiong@math.utk.edu
URL: http://www.math.utk.edu/~jxiong