DIFFUSIVE CLUSTERING OF INTERACTING BROWNIAN MOTIONS ON $\mathbb{Z}^2$

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Abstract. In this paper we investigate the cluster behavior of linearly interacting Brownian motions indexed by $\mathbb{Z}^2$. We show that (on a logarithmic scale) the block average process converges in *path space* to Brownian motion.

1. Introduction

Motivation. Many interacting particle systems in the d–dimensional space have the property that their long–time behavior is strongly dimension dependent with non–trivial equilibria in high dimensions (usually $d > 2$) and so–called clustering in low dimension (usually $d \leq 2$). In the critical dimension $d = 2$ some models have proved to show a phenomenon called *diffusive clustering*: a suitably defined block average process (zooming from large blocks to small blocks) converges to a diffusion process as time goes to infinity. Convergence of the finite dimensional marginals has been shown for a variety of models including: the voter model (Cox and Griffeath [CG86]), interacting diffusions on the hierarchical group (Fleischmann and Greven [FG94] and Klenke [Kle96] and branching Brownian motions and super–Brownian motion (Klenke [Kle97]) as well as for some infinite variance branching models (Klenke [Kle98]). However, so far for these models one could not establish convergence in path space.

Only recently Kopietz [Kop98] was able to show diffusive clustering and convergence in path space for a model of linearly interacting Brownian motions indexed by $\mathbb{Z}^2$. One aim of this paper is to give a more straightforward proof and to weaken the assumptions.

The method used in this paper heavily depends on the Gaussian structure of the process. However, there might be some hope that one can use this process as a prototype and employ comparison techniques to transfer the result to a broader class of interacting diffusions.

The model. Consider the following system of coupled stochastic differential equations

$$dx_t(i) = \sum_{j\in\mathbb{Z}^d} a(i,j)\big(x_t(j) - x_t(i)\big)\,dt \;+ dW_t(i), \qquad i \in \mathbb{Z}^d, \tag{1.1}$$

where $\{(W_t(i)_{t\geq 0},\ i \in \mathbb{Z}^d\}$ is an independent family of standard Brownian motions and a is the kernel of a random walk on $\mathbb{Z}^d$. We denote by $a^{(n)}$ its n–step transition

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probabilities. It is well–known that (to prove this one might proceed as in Shiga and Shimizu [SS80, proof of Theorem 3.2]) there exists a unique strong solution $(x_t)_{t\ge 0}$ of (1.1) taking values in a so–called Liggett–Spitzer space $X \subset \mathbb{R}^{\mathbb{Z}^d}$ (given that $x_0 \in X$). This space X is defined via the following procedure (see Liggett and Spitzer [LS81]). Fix a function $\gamma : \mathbb{Z}^d \to (0,\infty)$ with $\sum_{i\in\mathbb{Z}^d} \gamma(i) < \infty$ and the property

$$\sup_{i\in\mathbb{Z}^d} \frac{(\gamma a)(i)}{\gamma(i)} < \infty. \tag{1.2}$$

Such a function always exists. Define a norm $\|\cdot\|_\gamma$ by $\|x\|_\gamma = \sum_i |x(i)|\gamma(i)$ and let

$$X = \left\{ x \in \mathbb{R}^{\mathbb{Z}^d} : \ \|x\|_\gamma < \infty \right\}. \tag{1.3}$$

For example, if a is the kernel of simple random walk, then $\gamma(i) = (1 + \|i\|_2)^{-p}$ fulfills (1.2) for any $p > d$. Hence X is then a space with a polynomial growth condition.

Here we are interested in the longtime behavior of (x_t). It is known that the system clusters if the symmetrized kernel $\hat{a}(i,j) := \frac{1}{2}(a(i,j) + a(j,i))$ is recurrent. More precisely, if $x_0 \equiv 0$ then for all finite $A \subset \mathbb{Z}^d$ and all $K < \infty$

$$\lim_{t\to\infty} \mathbf{P}[x_t(i) \ge K,\ i \in A] = \lim_{t\to\infty} \mathbf{P}[x_t(i) \le -K,\ i \in A] = \frac{1}{2}. \tag{1.4}$$

This follows from a simple computation using the second moments (see, e.g., Cox and Klenke [CK99, Section 3.4]). Statement (1.4) remains true if we allow somewhat more general initial conditions, for example x_0 random with $\sup\{\mathbf{E}[|x_0(i)|],\ i \in \mathbb{Z}^d\} < \infty$. Apparently, (1.4) does not hold for infinite A. In fact, one can even show that almost surely

$$\limsup_{t\to\infty} x_t(i) = \infty \quad \text{and} \quad \liminf_{t\to\infty} x_t(i) = -\infty, \tag{1.5}$$

see [CK99].

One concept for investigating the clustering quantitatively is to introduce block averages. A block average value close to 0 indicates that many regions (clusters) with only positive or only negative coordinates compensate. The block is larger than the typical cluster. Large (absolute) values of the block average indicate that the block is covered by one cluster. Somewhere between these extremes one captures the typical size of a cluster. So one considers blocks of sizes that grow in time on different scales and tries to make a limit statement about the observed average values.

In this paper we shall focus on the situation where $d = 2$ and where a is the kernel of a non–degenerate, irreducible random walk with finite moments of order $(2+\delta)$ for some $\delta > 0$. We agree to denote by Q the covariance matrix of the symmetrized kernel $\hat{a}$, that is, Q is the 2×2–matrix associated with the quadratic form

$$\widetilde{Q}(y) = \sum_{k\in\mathbb{Z}^2} \hat{a}(0,k)\langle k, y\rangle^2, \qquad y \in \mathbb{R}^2, \tag{1.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product.

The rescaled block averages we consider are defined by

$$\theta_t(\alpha) = \left(\frac{2\pi\sqrt{\det Q}}{\log t}\right)^{1/2} \frac{1}{|B(t^{\alpha/2})|} \sum_{i\in B(t^{\alpha/2})} x_t(i), \qquad t>0,\ \alpha\in[0,1], \tag{1.7}$$

where

$$B(r) = \{i\in\mathbb{Z}^2 :\ \|i\|_\infty \le r\}. \tag{1.8}$$

Here α is a parameter that measures the scale at which the blocks grows in time. Our main result is the following theorem.

Theorem 1.1. *Assume that x_0 is ergodic with a $(2+\delta)$th moment for some $\delta > 0$. Let $(W_s)_{s\in[0,1]}$ be a Brownian motion. The process $(\theta_t(\alpha))_{\alpha\in[0,1]}$ converges as $t\to\infty$ in distribution in the Skorohod topology to $(W_{1-\alpha})_{\alpha\in[0,1]}$.*

It might be worthwhile noticing that the convergence is over $[0,1]$ not just $(0,1]$.

Remark 1.2. The condition on x_0 can be weakened, however this is not the main goal here. Note that x_t has an explicit representation

$$x_t(i) = a_t x_0(i) + \int_0^t a_{t-s}(i,j)dW_s(j), \qquad i\in\mathbb{Z}^2, \tag{1.9}$$

where

$$a_t(i,j) = e^{-t}\sum_{n=0}^{\infty}\frac{a^{(n)}(i,j)}{n!} \tag{1.10}$$

is the continuous time random walk kernel.

Hence once we have the statement of the theorem for $x_0\equiv 0$ we have it for any x_0 such that

$$\left((\log t)^{-1/2}|B(t^{\alpha/2})|^{-1}\sum_{i\in B(t^{\alpha/2})} a_t x_0(i)\right)_{\alpha\in[0,1]} \stackrel{t\to\infty}{\longrightarrow} ([0,1]\to\{0\}) \tag{1.11}$$

in the Skorohod topology. For the special choice in the theorem this follows from the ergodic theorem.

The difficult part in showing Theorem 1.1 is showing the tightness in the path space. To this end one usually has to compute fourth moments which in many cases is not so simple. Here, however, for $x_0\equiv 0$, everything is centered Gaussian and the fourth moments are a simple function of the variances.

Similar results as Theorem 1.1 have been obtained for the two–dimensional voter model, interacting diffusions, and spatial branching processes. However in these cases one has not been able to show convergence in path space but only convergence of the finite dimensional marginals.

Kopietz was the first to show the statement of Theorem 1.1, though under stronger assumptions, namely $(4+\delta)$th moments for x_0. In fact, Kopietz performs a painstaking direct computation of the fourth moments (this is, without using (1.9)) to obtain tightness in the Skorohod space.

In the next section we give the proof of Theorem 1.1 that makes use of some of the ideas in [Kop98].

2. Proof of Theorem 1.1

As indicated in Remark 1.2, it suffices to consider the case $x_0 \equiv 0$. In order to show the theorem we have to show

(i) convergence of the finite dimensional distributions,
(ii) tightness in the path space.

2.1. **Finite dimensional distributions.** Note that $(\theta_t(\alpha))_{\alpha\in[0,1]}$ is a Gaussian process, thus it is enough to show convergence of the covariance function

$$\Gamma_t(\alpha,\beta) = \mathbf{Cov}[\theta_t(\alpha),\theta_t(\beta)], \tag{2.1}$$

to the covariance function of $(W_{1-\alpha})_{\alpha\in[0,1]}$, namely

$$\lim_{t\to\infty}\Gamma_t(\alpha,\beta) = 1-(\alpha\vee\beta), \qquad \alpha,\beta\in[0,1]. \tag{2.2}$$

The key to (2.2) is formula (1.9) which yields immediately

$$\mathbf{Cov}[x_t(i),x_t(j)] = \frac{1}{2}\hat{G}_{2t}(i,j), \tag{2.3}$$

where $\hat{G}_t$ is the Green function of $\hat{a}$, that is

$$\hat{G}_t(i,j) = \int_0^t \hat{a}_s(i,j)\,ds = \sum_k \int_0^t a_{s/2}(i,k)a_{s/2}(k,j)\,ds. \tag{2.4}$$

Thus we have

$$\Gamma_t(\alpha,\beta) = \frac{2\pi\sqrt{\det Q}}{\log t}\Big(|B(t^{\alpha/2})|\cdot|B(t^{\beta/2})|\Big)^{-1}\sum_{\substack{i\in B(t^{\alpha/2})\\ j\in B(t^{\beta/2})}}\hat{G}_{2t}(i,j). \tag{2.5}$$

We introduce the function

$$A_t(i) = \hat{G}_t(0,0)-\hat{G}_t(0,i). \tag{2.6}$$

We will need the following statement about A_t that we prove below in Lemma 2.1

$$C := \sup_{t\geq 1}\sup_{\substack{i\in B(t^{\alpha/2})\\ i\neq 0}}\left|A_t(i)-\frac{1}{\pi\sqrt{\det Q}}\log\|i\|_\infty\right| < \infty. \tag{2.7}$$

Together with the well–known fact that (see Fukai and Uchiyama [FU96])

$$\lim_{t\to\infty}\frac{\hat{G}_t(0,0)}{\log t} = \frac{1}{2\pi\sqrt{\det Q}} \tag{2.8}$$

we get from (2.7) and (2.5)

$$\lim_{t\to\infty}\Gamma_t(\alpha,\beta) = 1-\lim_{t\to\infty}\frac{2\pi\sqrt{\det Q}}{\log t\cdot|B(t^{\alpha/2})|\cdot|B(t^{\beta/2})|}\sum_{\substack{i\in B(t^{\alpha/2}),\\ j\in B(t^{\beta/2}),\\ i\neq j}}\log\|i-j\|_\infty. \tag{2.9}$$

Using the fact that the overwhelming majority of points i,j have distance $\|i-j\|_\infty \approx t^{(\alpha\vee\beta)/2}$ we get

$$\lim_{t\to\infty}\Gamma_t(\alpha,\beta) = 1-(\alpha\vee\beta), \tag{2.10}$$

as desired.

2.2. **Tightness in path space.** A convenient sufficient condition for tightness of probability measures on the Skorohod space is (see [Bil68]): There exists a constant $\rho > 0$ and $t_0 > 0$ such that

$$\mathbf{E}[(\theta_t(\alpha) - \theta_t(\beta))^4] < \rho(\beta - \alpha)^2, \qquad t \geq t_0,\ \alpha, \beta \in [0, 1]. \tag{2.11}$$

Unfortunately, our function $\alpha \mapsto \theta_t(\alpha)$ has discontinuities at those points α where $t^{\alpha/2} \in \mathbb{N}$. Hence we cannot hope to verify (2.11) and to show tightness directly. However this is really only a minor problem: All we have to do is to change the definition of the block averages a little bit. In the sequel we agree to write for any function $f : \mathbb{Z}^2 \to \mathbb{R}$ and any $u \in \mathbb{R}^2$

$$f(u) = f(\lfloor u \rfloor), \tag{2.12}$$

where $\lfloor \cdot \rfloor$ is the Gauss bracket applied to both components. We also write $\widetilde{B}(r) = \{u \in \mathbb{R}^2 :\ \|u\|_\infty < r\}$. Now we define the *continuous block average* $\widetilde{\theta}_t(\alpha)$ by

$$\widetilde{\theta}_t(\alpha) = \frac{1}{4} t^{-\alpha} \left(\frac{2\pi\sqrt{\det Q}}{\log t} \right)^{1/2} \int_{\widetilde{B}(t^{\alpha/2})} du\ x_t(u). \tag{2.13}$$

It is simple to check that almost surely

$$\limsup_{t\to\infty} \sup_{\alpha\in[0,1]} \left| \widetilde{\theta}_t(\alpha) - \theta_t(\alpha) \right| = 0. \tag{2.14}$$

Hence it suffices to show tightness for $\theta_t(\alpha)$ by checking (2.11) for $\widetilde{\theta}_t(\alpha)$. Recall that the fourth moment of a centered Gaussian random variable Y can be expressed in terms its second moment by $\mathbf{E}[Y^4] = 3(\mathbf{E}[Y^2])^2$. Consequently all we have to show is that there exist $t_0 > 0$ and $\rho > 0$ such that

$$\mathbf{E}\big[(\widetilde{\theta}_t(\alpha) - \widetilde{\theta}_t(\beta))^2\big] \leq \rho|\beta - \alpha|, \qquad t \geq t_0,\ \alpha, \beta \in [0, 1]. \tag{2.15}$$

Define the covariance $\widetilde{\Gamma}_t(\alpha, \beta)$ anologously to (2.1). It is clear that $\alpha \mapsto \widetilde{\Gamma}_t(\alpha, \beta)$ is continuous and piecewise smooth. Hence in order to show (2.15) it will be enough to show that

$$\limsup_{t\to\infty} \sup_{\alpha,\beta\in[0,1]} \left| \frac{d}{d\alpha} \widetilde{\Gamma}_t(\alpha, \beta) \right| < \infty. \tag{2.16}$$

Note that

$$\widetilde{\Gamma}_t(\alpha, \beta) = \frac{\pi\sqrt{\det Q}}{4 \log t} t^{-\alpha-\beta} \int_{\widetilde{B}(t^{\alpha/2})} du \int_{\widetilde{B}(t^{\beta/2})} dv\ \hat{G}_{2t}(u, v). \tag{2.17}$$

Define the maps $f^i_{t,\alpha} : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3, 4$ by

$$\begin{aligned} f^1_{t,\alpha}(u_1, u_2) &= (u_1, t^{\alpha/2}), \\ f^2_{t,\alpha}(u_1, u_2) &= (u_1, -t^{\alpha/2}), \\ f^3_{t,\alpha}(u_1, u_2) &= (t^{\alpha/2}, u_2), \\ f^4_{t,\alpha}(u_1, u_2) &= (-t^{\alpha/2}, u_2). \end{aligned} \tag{2.18}$$

Then we can compute the derivative of $\widetilde{\Gamma}_t(\alpha,\beta)$ as

$$\begin{aligned}\frac{d}{d\alpha}\widetilde{\Gamma}_t(\alpha,\beta) =& \frac{\pi\sqrt{\det Q}}{4}t^{-\alpha-\beta}\\ &\times\sum_{i=1}^{4}\int_{\widetilde{B}(t^{\alpha/2})} du \int_{\widetilde{B}(t^{\beta/2})} dv\ [\hat{G}_{2t}(f^i_{t,\alpha}(u),v) - \hat{G}_{2t}(u,v)].\end{aligned} \tag{2.19}$$

Recalling (2.7) we get that there exist constants C' and C'' such that

$$\left|\frac{d}{d\alpha}\widetilde{\Gamma}_t(\alpha,\beta)\right| \le C' + 4C'' I_t(\alpha,\beta), \tag{2.20}$$

where

$$I_t(\alpha,\beta) = t^{-\alpha-\beta}\int_{\widetilde{B}(t^{\alpha/2})} du \int_{\widetilde{B}(t^{\beta/2})} dv \left|\log\frac{\|f^1_{t,\alpha}(u)-v\|_2}{\|u-v\|_2}\right|. \tag{2.21}$$

It is an exercise to check that $I_t(\alpha,\beta) \le 100$ (see Lemma 2.2 below). Hence we have shown that

$$\left|\frac{d}{d\alpha}\widetilde{\Gamma}_t(\alpha,\beta)\right| \le C' + 400C'' \qquad \text{for all } t \ge 1, \alpha,\beta \in [0,1]. \tag{2.22}$$

This completes the proof of the tightness and thus of Theorem 1.1

It remains to state and show the two lemmas that we made use of in the preceding proof.

Lemma 2.1. *For all $K > 0$ there exists a constant C such that for all $t \ge 1$ and $k \in \mathbb{Z}^2$, $k \ne 0$, $\|k\|_\infty < Kt^{1/2}$,*

$$\left|A_t(k) - \frac{1}{\pi\sqrt{\det Q}}\log\|k\|_\infty\right| \le C. \tag{2.23}$$

Proof. It is well-known that (see [Spi76])

$$A(k) = \lim_{t\to\infty} A_t(k), \qquad k \in \mathbb{Z}^2, \tag{2.24}$$

exists. A is called the *recurrent potential kernel.* In [FU96, Theorem 1] it is shown that there exists a constant c_0 such that

$$\lim_{\|k\|_\infty\to\infty}\left|A(k) - \frac{1}{\pi\sqrt{\det Q}}\log\|k\|_Q - c_0\right| = 0, \tag{2.25}$$

where $\|\cdot\|_Q$ is the norm on $\mathbb{R}^2$ defined by $\|u\|_Q = \sqrt{u^T Q^{-1} u}$. Using the equivalence of norms, this implies

$$\sup_{k\in\mathbb{Z}^2}\left|A(k) - \frac{1}{\pi\sqrt{\det Q}}\log\|k\|_\infty\right| < \infty. \tag{2.26}$$

We are done if we can show that for some $c > 0$

$$A(k) - A_t(k) \le c\frac{\|k\|_\infty}{\sqrt{t}}, \qquad t \ge 1,\ k \in \mathbb{Z}^2. \tag{2.27}$$

We proceed similarly as in [Spi76] for the proof of the existence of A. Let ϕ be the characteristic function of $\hat{a}$, that is

$$\phi(\theta) = \sum_{k\in\mathbb{Z}^2}\exp(i\langle\theta,k\rangle)\hat{a}(0,k), \qquad \theta \in \mathbb{R}^2. \tag{2.28}$$

Thus, using the Fourier inversion formula, we have

$$\hat{a}_t(0,k) = (2\pi)^{-2} \int_{[-\pi,\pi]^2} d\theta\, e^{-i\langle\theta,k\rangle} e^{t(\phi(\theta)-1)}. \tag{2.29}$$

Thus

$$\begin{aligned} A(k) - A_t(k) &= \int_t^\infty ds\, \hat{a}_s(0,0) - \hat{a}_s(0,k) \\ &= (2\pi)^{-2} \int_{[-\pi,\pi]^2} d\theta \int_t^\infty ds\, \left(1 - e^{-i\langle\theta,k\rangle}\right) e^{s(\phi(\theta)-1)} \\ &= (2\pi)^{-2} \int_{[-\pi,\pi]^2} d\theta\, \frac{1 - e^{-i\langle\theta,k\rangle}}{1-\phi(\theta)} e^{t(\phi(\theta)-1)}. \end{aligned} \tag{2.30}$$

Now we make use of the fact that (see [Spi76, Proposition 7.5]) there exists a constant $\lambda > 0$ such that the real part of $1-\phi(\theta)$ is larger than $\lambda\|\theta\|_2^2$, $\theta \in [-\pi,\pi]^2$. Hence we get

$$\begin{aligned} A(k) - A_t(k) &\le \frac{\|k\|_2}{(2\pi)^2\lambda} \int_{\mathbb{R}^2} d\theta\, \frac{e^{-\lambda t\|\theta\|_2^2}}{\|\theta\|_2} \\ &= \frac{1}{2\sqrt{\pi}\lambda^{3/2}} \frac{\|k\|_2}{\sqrt{t}}. \end{aligned} \tag{2.31}$$

Set $c = \sqrt{2/\pi}\lambda^{-3/2}$ to conclude (2.27). □

Finally we show the following lemma. Recall the definition of $I_t(\alpha,\beta)$ in (2.21).

Lemma 2.2. *For all $t \ge 1$ and $\alpha, \beta \in [0,1]$*

$$I_t(\alpha,\beta) \le 100. \tag{2.32}$$

Proof. Note that, with $T = t^{(\beta-\alpha)/2}$,

$$I_t(\alpha,\beta) = \frac{1}{2}\int_{[-1,1]^2} du \int_{[-1,1]^2} dv\, \left|\log \frac{\|(u_1,1) - Tv\|_2^2}{\|u - Tv\|_2^2}\right|. \tag{2.33}$$

Consider first the case $T \le 1$. Here

(2.34)

$$\begin{aligned} I_t(\alpha,\beta) &\le -\int_{[-1,1]^2} du \int_{[-1,1]^2} dv\, \left[\log\left(\frac{1}{8}\|(u_1,1) - Tv\|_2^2\right) + \log\left(\frac{1}{8}\|u - Tv\|_2^2\right)\right] \\ &\le -4 \int_{-1}^1 du_1 \int_{-1}^1 dv_1 \log\left(\frac{1}{8}(u_1 - Tv_1)^2\right) \\ &\le 16 \int_0^2 du_1\, \log\left(\frac{1}{8}u^2\right) \\ &= 32(2 + \log 2) \le 100. \end{aligned}$$

For $T > 1$ the same estimate yields

$$\begin{aligned} I_t(\alpha,\beta) &\le -4 \int_{-1}^1 du_1 \int_{-1}^1 dv_1 \log\left(\frac{1}{8}(T^{-1}u_1 - v_1)^2\right) \\ &\le 32(2+\log 2) \le 100. \end{aligned} \tag{2.35}$$

This finishes the proof. □

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